A Tableau for RTL (long version)

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Abstract

We use mosaics to provide a simple, sound, complete and terminating tableau reasoning procedure for the temporal logic of until and since over the real numbers model of time.

1 Introduction

Although discrete time temporal logics are the most common, there has been a separate thread of steady development of continuous time alternatives since the earliest beginnings. Being able to reason about events and processes unfolding continuously has an enormous range of applications from concurrency and refinement in reactive systems, as a basis for the metric temporal logics used for model checking automated systems, to artificial planning, natural language semantics and philosophical arguments.

In this paper we investigate the most natural and useful such temporal logic: RTL, the propositional temporal logic over real-numbers time using the Until and Since connectives introduced in [Kam68]. RTL is as expressive as first-order logics over linear structures [Kam68]. It is decidable [BG85, Rey10] (in PSPACE) and has complete axioms systems [GH90, Rey92].

Currently there is no satisfiability or validity checking procedure for RTL that looks remotely amenable to implementation. In this paper we present what seems to be an intuitive tableau style decision procedure for RTL which will not be hard to implement.

The proof of correctness here uses the mosaics which were used to prove PSPACE decidability of RTL in [Rey10]. Mosaics are small pieces of a model. We can decide whether a finite set of mosaics is sufficient to be used to build a real-numbers model of a given formula by considering something like a game tree which can also be viewed as a tableau. Such an idea was suggested for general dense time reasoning in [Rey09] but here we have to look carefully at shapes of sub-graphs within the tree to enforce the peculiar properties of the reals: such as density, Dedekind completeness and separability.
The contribution here are as follows. Presenting a sound and complete mosaic-based tableau system to decide satisfiability in RMS. We aim mainly to show clearly how mosaics can be the building blocks of a tableau with this logic here: the system is not at all streamlined and is intended to provide the foundation of more intelligent tableau construction techniques in future work.

Below we define the logic RTL in section 2, explain mosaics in section 3, show how to make a sufficient set of mosaics in section 5, lay out the basic mosaic tableau in section 6, adjust it for the case of the reals flow in Section 7, soundness in Section 8 and finally prove completeness in Section 9.

2 The logic

Fix a countable set \( L \) of atoms. Here, frames \( (T, <) \), or flows of time, will be irreflexive linear orders. Structures \( T = (T, <, h) \) will have a frame \( (T, <) \) and a valuation \( h \) for the atoms, i.e. for each atom \( p \in L, h(p) \subseteq T \). Of particular importance will be real structures \( T = (\mathbb{R}, <, h) \), which have the real numbers flow (with their usual irreflexive linear ordering).

The language \( L(U, S) \) is generated by the 2-place connectives \( U \) and \( S \) along with classical \( \neg \) and \( \land \). That is, we define the set of formulas recursively to contain the atoms and for formulas \( \alpha \) and \( \beta \) we include \( \neg \alpha \), \( \alpha \land \beta \), \( U(\alpha, \beta) \) and \( S(\alpha, \beta) \).

Formulas are evaluated at points in structures \( T = (T, <, h) \). We write \( T, x \models \alpha \) when \( \alpha \) is true at the point \( x \in T \). This is defined recursively as follows. Suppose that we have defined the truth of formulas \( \alpha \) and \( \beta \) at all points of \( T \). Then for all points \( x \):

\[
\begin{align*}
T, x & \models p \quad \text{iff} \quad x \in h(p), \text{ for } p \text{ atomic;} \\
T, x & \models \neg \alpha \quad \text{iff} \quad T, x \not\models \alpha; \\
T, x & \models \alpha \land \beta \quad \text{iff} \quad \text{both } T, x \models \alpha \text{ and } T, x \models \beta; \\
T, x & \models U(\alpha, \beta) \quad \text{iff} \quad \text{there is } y > x \text{ in } T \text{ such that } T, y \models \alpha \\
& \quad \text{and for all } z \in T \text{ such that } x < z < y, \text{ we have } T, z \models \beta; \text{ and} \\
T, x & \models S(\alpha, \beta) \quad \text{iff} \quad \text{there is } y < x \text{ in } T \text{ such that } T, y \models \alpha \\
& \quad \text{and for all } z \in T \text{ such that } y < z < x, \text{ we have } T, z \models \beta.
\end{align*}
\]

Often, definitions, results or proofs will have a mirror image in which \( U \) and \( S \) are exchanged and \( < \) and \( > \) swapped. We do not always mention the mirror image.

In most of the literature on temporal logics for discrete time, the “until” connective is written in an infix manner: \( \beta U \alpha \) rather than \( U(\alpha, \beta) \). This corresponds to the natural language reading “I will be here until I become hungry” rather than our alternative “until I am hungry, I will be
We choose to use the prefix notation for until (and since) because it agrees with important previous work on the language for dense time such as [Kam68], [BG85] and [GHR94] and because the infix until connective seen with discrete time is usually a slightly different connective, the non-strict until connective which we mention below.

2.1 Abbreviations

For the main proof in this paper we do not need any of the many common and generally useful other connectives which can be defined as abbreviations in the language. In fact, they may be distracting here. However, they are often useful in applications, they are interesting, they do help us with a few illustrative examples below and they are useful in presenting several of the secondary results below. So, for the sake of neatness in collecting them in one place, we present a catalogue here.

There are the classical \( \alpha \lor \beta = \neg(\neg \alpha \land \neg \beta) \); \( \top = p \lor \neg p \) (where \( p \) is some particular atom from \( L \)); \( \bot = \neg \top \); and \( \alpha \rightarrow \beta = (\neg \alpha) \lor \beta \).

Then there are the common temporal ones: \( F\alpha = U(\alpha, \top) \), “alpha will be true (sometime in the future)” ; \( G\alpha = \neg F(\neg \alpha) \), “alpha will always hold (in the future)” ; and their mirror images \( P \) and \( H \). Particularly for dense time applications we also have: \( C^+\alpha = U(\top, \alpha) \), “alpha will be constantly true for a while after now”; and \( K^+\alpha = \neg C^+\neg \alpha \), “alpha will be true arbitrarily soon”. They have mirror images \( C^- \) and \( K^- \).

For discrete flows of time such as the natural numbers on which the useful PLTL logic is defined, a “next-time” or “tomorrow” connective is often used. This can be defined more generally over any linear order via \( X\alpha = U(\alpha, \bot) \). The mirror image is \( Y\alpha = S(\alpha, \bot) \), sometimes called “strong yesterday” and there is also a “weak yesterday” connective which abbreviates \( \neg Y\neg \alpha \).

The non-strict “until” connective [SC85], used in PLTL and other temporal logics over the natural numbers (i.e. over sequences of states) is just “\( \alpha \) until \( \beta \)” given as \( \beta \lor (\alpha \land U(\beta, \alpha)) \) in terms of our strict “until”. There is a mirror image non-strict “since”. Comparisons between strict and non-strict connectives are discussed more fully in [Rey03].

2.2 Reasoning with RTL

A formula \( \phi \) is \( \mathbb{R} \)-satisfiable if it has a real model: i.e. there is a real structure \( S = (\mathbb{R}, <, h) \) and \( x \in \mathbb{R} \) such that \( S, x \models \phi \). A formula is \( \mathbb{R} \)-valid iff it is true at all points of all real structures. Of course, a formula is \( \mathbb{R} \)-valid iff its negation is not \( \mathbb{R} \)-satisfiable. We will refer to the logic of \( L(U,S) \) over real structures as RTL.

Let RTL-SAT be the problem of deciding whether a given formula of \( L(U,S) \) is \( \mathbb{R} \)-satisfiable or not. [Rey10] proves:

**Theorem 1** RTL-SAT is PSPACE-complete.
In order to help get a feel for the sorts of formulas which are validities in RTL it is worth considering a few formulas in the language. \( U(\top, \bot) \) is a formula which only holds at a point with a discrete successor point so \( G \neg U(\top, \bot) \) is a validity of RTL. \( Fp \rightarrow FFp \) is a formula which can be used as an axiom for density and is also a validity of RTL.

**DEFINITION 2** A linear order is Dedekind complete if and only if each non-empty subset which has an upper bound has a least upper bound.

\[(\Gamma^p \land F\neg p) \rightarrow U(\neg p \lor K^+ (\neg p), p)\] was used as an axiom for Dedekind completeness (in [Rey92]) and is a validity. The formula says that if \( p \) is true constantly for a while but not forever then there is an upper bound on the interval in which it remains true. This formula is not valid in the temporal logic with until and since over the rational numbers flow of time.

One of the most interesting validities of RTL is Hodkinson’s axiom “Sep” (see [Rey92]). It is

\[K^+(p \land \neg K^+(p \land U(p, \neg p))) \rightarrow K^+(K^+(p \land K^-(p))).\]

This can be used in an axiomatic completeness proof to enforce the separability of the linear order:

**DEFINITION 3** A linear order is separable if and only if it has a countable suborder which is spread densely throughout the order: i.e. between every two elements of the order lies an element of the suborder.

The fact that the rationals are spread densely throughout the reals, shows that the reals are separable. There are dense, Dedekind complete linear orders with end points which are not separable (e.g. see [Rey92]). The negation of Sep will be satisfiable in such flows but not over the reals.\(^1\)

As we have noted in the introduction, there are complete axiom systems for RTL in [GH90] and in [Rey92]: the former using a special rule of inference and the latter just using orthodox rules.

For the interested reader we briefly outline the axiom system presented in [Rey92]. It has the usual rules for a temporal logic: i.e. modus ponens, past and future temporal generalizations and substitution. The axioms are:

- all classical tautologies,
- the six Burgess-Xu axioms

\[G(p \rightarrow q) \rightarrow (U(p, r) \rightarrow U(q, r))\]

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\(^1\)Note, however, that it is not the case that Sep is valid in precisely the separable orders. The long line \([0, 1] \times \omega_1\) consisting of \(\omega_1\) copies of \([0, 1]\) laid end to end is dense, Dedekind complete, and not separable and yet Sep is valid in it.
\[ G(p \rightarrow q) \rightarrow (U(r, p) \rightarrow U(r, q)) \]

\[ p \land U(q, r) \rightarrow U(q \land S(p, r), r) \]

\[ U(p, q) \rightarrow U(p, q \land U(p, q)) \]

\[ U(q \land U(p, q), q) \rightarrow U(p, q) \]

\[ U(p, q) \land U(r, s) \rightarrow \]

\[ U(p \land r, q \land s) \lor U(p \land s, q \land s) \lor U(q \land r, q \land s) \]

along with each of their duals,

- plus axioms for density and no end points:
  \( K^+ \top, K^− \top, F \top \) and \( P \top \)

- two for Dedekind completeness:
  \( \text{Prior-U: } U(\top, p) \land F \neg p \rightarrow U(\neg p \lor K^+(\neg p), p) \)
  \( \text{Prior-S: } S(\top, p) \land P \neg p \rightarrow S(\neg p \lor K^−(\neg p), p) \)

- and Sep:
  \( \text{Sep: } K^+ p \land \neg K^+(p \land U(p, \neg p)) \rightarrow K^+(K^+ p \land K^− p) \)

Rabin’s decision procedure for the second-order monadic logic of two successors [Rab69] is used in [BG85] to show that that RTL is decidable. One of the two decision procedures in that paper just gives us a non-elementary upper bound on the complexity of RTL-SAT.

There seems to have been little further development of any reasoning techniques for RTL. Standard techniques for temporal reasoning including automata, tableaux, finite model properties and resolution do not seem to give any easy answers. For example, automata operate with discrete steps. The main general problem, though, is that all the usual ways of using these techniques are based on reasoning about what is true at one point at a time.

Deciding RTL seems to need the ability to reason about (at least) two points and the points in between them. This is exactly the way that the new mosaic technique for temporal logic has been seen to work in [Rey03] where it was applied to the temporal logic with \( U \) over the class of all linear orders.

Thus we launch into using the new mosaic technique for the more complicated and more useful specific case of the real numbers flow of time. With general ways of developing tableaux from mosaics suggested in [MMR00], it might then be possible to portray the procedure in a more standard way: but that is future work.

3 Mosaics for \( U \) and \( S \)

We will decide the satisfiability of formulas by considering sets of simple labelled structures which represent small pieces of real structures. The idea
is based on the mosaics seen in [N95] and used in many other subsequent proofs.

Each mosaic is a small piece of a model, i.e. a small set of objects (points), relations between them and a set of formulas for each point indicating which formulas are true there in the whole model. There will be coherence conditions on the mosaic which are necessary for it to be part of a larger model.

We want to show the equivalence of the existence of a model to the existence of a certain set of mosaics: enough mosaics to build a whole model. So the whole set of mosaics also has to obey some conditions. These are called saturation conditions. For example, a particular small piece of a model might require a certain other piece to exist somewhere else in the model. We talk of the first mosaic having a defect which is cured by the latter mosaic.

Our mosaics will only be concerned with a finite set of formulas:

**DEFINITION 4** For each formula \( \phi \), define the closure of \( \phi \) to be 
\[
\text{Cl} \phi = \{ \psi, \neg \psi \mid \psi \leq \phi \}
\]
where \( \chi \leq \psi \) means that \( \chi \) is a subformula of \( \psi \).

We can sometimes think of \( \text{Cl} \phi \) as being closed under negation: treat \( \neg \neg \alpha \) as if it was \( \alpha \).

Often we will intend that a set of formulas will be exactly the set of formulas which hold at a particular point in a model. Such a set should at least be consistent in terms of classical propositional logic:

**DEFINITION 5** Suppose \( \phi \in L(U, S) \) and \( S \subseteq \text{Cl} \phi \). Say \( S \) is propositionally consistent (PC) iff there is no substitution instance of a tautology of classical propositional logic of the form \( \neg (\alpha_1 \land \ldots \land \alpha_n) \) with each \( \alpha_i \in S \). Say \( S \) is maximally propositionally consistent (MPC) iff \( S \) is maximal in being a subset of \( \text{Cl} \phi \) which is PC.

We will define a mosaic to be a triple \((A, B, C)\) of sets of formulas. The intuition is that this corresponds to two points from a structure: \( A \) is the set of formulas (from \( \text{Cl} \phi \)) true at the earlier point, \( C \) is the set true at the later point and \( B \) is the set of formulas which hold at all points strictly in between. Look ahead to definition 16 to see how mosaics can be found in a real structure.

The coherency conditions are given as part of the following definition. It will be easy to see that they are necessary for a mosaic to represent a small part of a real structure. However, they are only simple syntactic criteria and are therefore not subtle enough to be also sufficient for a mosaic to represent a piece of real structure. Thus, as we will see later, an important task in this paper is to identify which mosaics are actually realizable.

**DEFINITION 6** Suppose \( \phi \) is from \( L(U, S) \). A \( \phi \)-mosaic is a triple \((A, B, C)\) of subsets of \( \text{Cl} \phi \) such that:
C0.1 A and C are maximally propositionally consistent, and
C0.2 for all $\beta \in \text{cl} \phi$ with $\neg \neg \beta \in \text{cl} \phi$ we have $\neg \neg \beta \in B$ iff $\beta \in B$ and the following four coherence conditions hold:

C1. if $\neg U(\alpha, \beta) \in A$ and $\beta \in B$ then we have both:
   C1.1. $\neg \alpha \in C$ and either $\neg \beta \in C$ or $\neg U(\alpha, \beta) \in C$; and
   C1.2. $\neg \alpha \in B$ and $\neg U(\alpha, \beta) \in B$.

C2. if $U(\alpha, \beta) \in A$ and $\neg \alpha \in B$ then we have both:
   C2.1. either $\alpha \in C$ or both $\beta \in C$ and $U(\alpha, \beta) \in C$; and
   C2.2. $\beta \in B$ and $U(\alpha, \beta) \in B$.

C3-4 mirror images of C1-C2.

When dealing with mosaics $(A, B, C)$, the act of mirror imaging also involves swapping $A$ and $C$. Thus, to help make the concept of mirror images abundantly clear we will, for once, spell out the mirror imaging:

C3. if $\neg S(\alpha, \beta) \in C$ and $\beta \in B$ then we have both:
   C3.1. $\neg \alpha \in A$ and either $\neg \beta \in A$ or $\neg S(\alpha, \beta) \in A$; and
   C3.2. $\neg \alpha \in B$ and $\neg S(\alpha, \beta) \in B$.

C4. if $S(\alpha, \beta) \in C$ and $\neg \alpha \in B$ then we have both:
   C4.1. either $\alpha \in A$ or both $\beta \in A$ and $S(\alpha, \beta) \in A$; and
   C4.2. $\beta \in B$ and $S(\alpha, \beta) \in B$.

The reader can check that these coherence conditions are reasonable (i.e. sound) in terms of the intended meaning of a mosaic. For example, considering C2.2, if $U(\alpha, \beta)$ holds at one point $x$ and $\neg \alpha$ holds at all points between $x$ and $y > x$, then it is clear from the semantics of $U$ that there must be some $z \geq y$ with $\alpha$ true there and $\beta$ (and so also $U(\alpha, \beta)$) holding everywhere between $x$ and $y$ and beyond until $z$.

**DEFINITION 7** If $m = (A, B, C)$ is a mosaic then $\text{start}(m) = A$ is its start, $\text{cover}(m) = B$ is its cover and $\text{end}(m) = C$ is its end.

If we start to build a model using mosaics then, as we have noted, we may realize that the inclusion of one mosaic necessitates the inclusion of others: defects need curing. If we claim to have in a certain set all the mosaics needed to build a model, i.e. we have a saturated set of mosaics, then the other mosaics should be in our set too. For example,—this is 1.2 below—if we have $U(\alpha, \beta)$ holding at $x < y$ and neither $\alpha$ nor $\beta$ true at $y$ then it is clear that there is a point $z$ with $x < z < y$, $\alpha$ true at $z$ and $\beta$ true everywhere between $x$ and $z$. If there is such a point $z$ and we claim to have a saturated set of mosaics then we should have the mosaics corresponding to the pairs $(x, z)$ and $(z, y)$ as well as $(x, y)$. Below we will see that we cure defects en masse via a whole sequence of other mosaics rather than just having a pair to cure one defect at a time as in this example.

**DEFINITION 8** A defect in a mosaic $(A, B, C)$ is either
1. a formula $U(\alpha, \beta) \in A$ with either
   1.1 $\beta \notin B$,
   1.2 ($\alpha \notin C$ and $\beta \notin C$), or
   1.3 ($\alpha \notin C$ and $U(\alpha, \beta) \notin C$);
2. a formula $S(\alpha, \beta) \in C$ with either
   2.1 $\beta \notin B$,
   2.2 ($\alpha \notin A$ and $\beta \notin A$), or
   2.3 ($\alpha \notin A$ and $S(\alpha, \beta) \notin A$); or
3. a formula $\beta \in \text{Cl}_\phi$ with $\neg \beta \notin B$.

We refer to defects of type 1 to 3 (as listed here). Note that the same formula may be both a type 1 or 2 defect and a type 3 defect in the same mosaic. In that case we count it as two separate defects.

A little careful reasoning with several forms of formulas gives us the following:

**LEMMA 9** If $m$ is a mosaic and $\beta \in \text{Cl}_\phi \setminus \text{cover}(m)$ then $\neg \beta$ is a type 3 defect in $m$.

We will need to string mosaics together to build linear orders. This can only be done under certain conditions. Here we introduce the idea of composition of mosaics.

**DEFINITION 10** We say that $\phi$-mosaics $(A', B', C')$ and $(A'', B'', C'')$ compose if $C' = A''$. In that case, their composition is $(A', B' \cap C' \cap B'', C'')$.

It is straightforward to prove that this is a mosaic and that composition of mosaics is associative.

**LEMMA 11** If mosaics $m$ and $m'$ compose then their composition is a mosaic.

**LEMMA 12** Composition of mosaics is associative.

Thus we can talk of sequences of mosaics composing and then find their composition. We define the composition of a sequence of length one to be just the mosaic itself. We leave the composition of an empty sequence undefined.

**DEFINITION 13** A decomposition for a mosaic $(A, B, C)$ is any finite sequence of mosaics $(A_1, B_1, C_1), (A_2, B_2, C_2), \ldots, (A_n, B_n, C_n)$ which composes to $(A, B, C)$.

It will be useful to introduce an idea of fullness of decompositions. This is intended to be a decomposition which provides witnesses to the cure of every defect in the decomposed mosaic.
DEFINITION 14 The decomposition above is full iff the following three conditions all hold:

1. for all $U(\alpha, \beta) \in A$ we have
   1.1. $\beta \in B$ and either $(\beta \in C$ and $U(\alpha, \beta) \in C$) or $\alpha \in C$,
   1.2. or there is some $i$ with $1 \leq i < n$ such that $\alpha \in C_i$, $\beta \in B_j$ (all $j \leq i$) and $\beta \in C_j$ (all $j < i$);
2. the mirror image of 1.; and
3. for each $\beta \in Cl\phi$ such that $\neg \beta \not\in B$ there is some $i$ such that $1 \leq i < n$ and $\beta \in C_i$.

If 1.2 above holds in the case that $U(\alpha, \beta) \in A$ is a type 1 defect in $(A, B, C)$ then we say that a cure for the defect is witnessed (in the decomposition) by the end of $(A_i, B_i, C_i)$. Similarly for the mirror image for $S(\alpha, \beta) \in C$. If a cure for any defect is witnessed then we say that the defect is cured.

LEMMA 15 If $m_1, ..., m_n$ is a full decomposition of $m$, then every defect in $m$ is cured in the decomposition.

For the reals we do not allow full decompositions of length one, although they are allowed in general linear time contexts for mosaics with no defects.

4 Satisfiability and relativization

In this section we define a notion of satisfiability for mosaics and relate the satisfiability of formulas (which is our ultimate interest) to that of mosaics.

Because mosaics represent linear orders with end points, it is inconvenient for us to continue to work directly with $\mathbb{R}$ and because we want to make use of some simple tricks with convergence of sequences in the metric at several places in the proof, we will move to work in the unit interval $[0, 1]$ instead.

If $x < y$ from $\mathbb{R}$ then let $]x, y[\}$ denote the open interval $\{z \in \mathbb{R} | x < z < y\}$ and $[x, y]$ denote the closed interval $\{z \in \mathbb{R} | x \leq z \leq y\}$. Similarly for half open intervals.

One can get a mosaic from any two points in a structure.

DEFINITION 16 If $T = (T, <, h)$ is a structure and $\phi$ a formula then for each $x < y$ from $T$ we define $mos_T(x, y) = (A, B, C)$ where:

$A = \{\alpha \in Cl\phi | x \models \alpha\}$,
$B = \{\beta \in Cl\phi | for all z \in T, if x < z < y then T, z \models \beta\}$, and
$C = \{\gamma \in Cl\phi | y \models \gamma\}$. 
It is straightforward to show that this is a mosaic.

**Lemma 17** \( \text{mos}^\phi_T(x,y) \) is a mosaic.

If \( T \) and \( \phi \) are clear from context then we just write \( \text{mos}(x,y) \) for \( \text{mos}^\phi_T(x,y) \).

**Definition 18** Suppose \( T \subseteq \mathbb{R} \). Let \( < \) also denote the restriction of \( < \) to any such \( T \). We say that a \( \phi \)-mosaic \( m \) is \( T \)-satisfiable iff there is some structure \( T = (T,<,h) \) such that \( m = \text{mos}^\phi_T(x,y) \) for some \( x < y \) from \( T \).

**Definition 19** We say that a \( \phi \)-mosaic is fully \([0,1]\)-satisfiable iff it is \( \text{mos}^\phi_T(0,1) \) for some structure \( T = ([0,1],<,h) \).

Next, we say a little more about satisfiable mosaics and amongst other things, that the set of all such mosaics is closed under composition. These results will be needed in the main lemma later in the proof but they may also help the reader in developing an intuitive idea of the mosaic concept.

**Lemma 20** Suppose that \( \psi \in L(U,S) \) and that \( m = (A,B,C) \) is \([0,1]\)-satisfiable.

Then \( |B| \) is at most the length of \( \psi \).

**Proof:** To see this, note that \( \psi \) will have at most \( L = |\psi| \) subformulas. If \( B \) contains at most one of \( \phi \) or \( \neg \phi \) for each of these \( \leq L \) subformulas \( \phi \) then \( B \) contains at most \( L \) formulas.

If \( B \) contains both \( \phi \) and \( \neg \phi \) then it is clear that \( m \) cannot be \( \text{mos}(x,y) \) for any \( x < y \) from \([0,1]\) and so cannot be satisfiable: just consider how \( \phi \) and \( \neg \phi \) can both hold at \((x+y)/2\). \( \square \)

**Lemma 21** Suppose that we have \( \psi \in L(U,S) \) and that \( m \) and \( n \) are \([0,1]\)-satisfiable \( \psi \)-mosaics which compose.

Then their composition is also \([0,1]\)-satisfiable.

Furthermore there is a model of both \( m \) and \( n \) with the mosaics adjacent.

**Lemma 22** Suppose that we have \( \psi \in L(U,S) \), a structure \( S \), \( 0 \leq x < y \leq 1 \) and \( m \) is the \([0,1]\)-satisfiable \( \psi \)-mosaic \( \text{mos}_S(x,y) \).

Suppose also that \( 0 \leq u < v \leq 1 \) such that:

\[
0 = u \quad \text{iff} \quad x = 0 \quad \text{and} \quad v = 1 \quad \text{iff} \quad y = 1.
\]

Then there is a structure \( T \) such that \( m = \text{mos}_T(u,v) \).

**Proof:** This is straightforward using any one-to-one, onto, order-preserving map \( \mu : [0,1] \rightarrow [0,1] \) such that \( \mu(u) = x \) and \( \mu(v) = y \). Define \( T = ([0,1],<,h) \) from \( S = ([0,1],<,g) \) by \( t \in h(p) \) iff \( \mu(t) \in g(p) \). Use an induction on the construction of \( \alpha \) to show that for all \( \alpha \in \text{Cl}(\phi) \), for all \( t \in [0,1] \), \( T,t \models \alpha \) iff \( S,\mu(t) \models \alpha \). Then \( \text{mos}_T(u,v) = \text{mos}_S(x,y) \). \( \square \)
DEFINITION 23 Suppose that we have $\psi \in L(U, S)$, and a $[0, 1]$-satisfiable $\psi$-mosaic $m$.

Then we say that $m$ is initially satisfiable iff there is a structure $S$ and $0 < y \leq 1$ such that $m = \text{mos}_S(0, y)$.

We say that $m$ is finally satisfiable iff there is a structure $S$ and $0 \leq x < 1$ such that $m = \text{mos}_S(x, 1)$.

The next lemma is useful as it allows us to build a structure in a piece-meal way from parts of structures which each satisfy a mosaic from a composing sequence of mosaics. Furthermore, the mosaics are still satisfied in the constructed structure.

LEMMA 24 Suppose that we have $\psi \in L(U, S)$, $n \geq 1$, a sequence $0 = x_0 < x_1 < \ldots < x_n = 1$ and a sequence $m_1, m_2, \ldots, m_n$ of $[0, 1]$-satisfiable $\psi$-mosaics such that:

1. $m_1$ is initially satisfiable;
2. $m_n$ is finally satisfiable; and
3. the mosaics compose, i.e. for each $i = 1, \ldots, n - 1$,
   \begin{align*}
   \text{end}(m_i) &= \text{start}(m_{i+1}).
   \end{align*}

Then, there is a structure $T$, such that for each $i = 0, 1, \ldots, n - 1$,

$\text{mos}_T(x_i, x_{i+1}) = m_{i+1}$.

Furthermore, if there is $I \subseteq \{1, \ldots, n\}$, and structures $S_i$ based on $[0, 1]$ such that each $m_i = \text{mos}_{S_i}(x_{i-1}, x_i)$, then we can assume that for each $i \in I$, $S_i$ and $T$ agree on the truth of all formulas in $\text{Cl} (\psi)$ at all points in $[x_{i-1}, x_i]$.

PROOF: Given the $x_i$ and $m_i$ we can, via lemma 22, suppose that $m_i = \text{mos}_{S_i}(x_i, x_{i+1})$ for some structures $S_i = ([0, 1], <, h_i)$ which are already given to us when $i \in I$. Note that we may use lemma 21 to deduce that each $m_i$ for $i \notin I$ and $2 \leq i \leq n - 1$ is satisfiable in the interior $[0, 1]$.

Say that each $S_i = ([0, 1], <, h_i)$. Now let $T = ([0, 1], <, h)$ where $t \in h(p)$ iff there is $i$ such that $x_i \leq t < x_{i+1}$ or $t = 1$ and $i = n$ and $t \in h_i(p)$.

We can use a straightforward induction on the construction of $\alpha$ to show that for all $i$, for all $\alpha \in \text{Cl}(\psi)$, for all $t \in [x_{i-1}, x_i]$, $T, t \models \alpha$ iff $S_i, t \models \alpha$. The more interesting cases of $U$ and $S$ are similar to the case of $U$ in lemma 21. □

We will now relate the satisfiability of a formula $\phi$ to that of certain mosaics. Obviously, a formula will be satisfiable over the reals iff it is satisfiable over the $[0, 1]$ flow. Furthermore, this happens iff a relativized version of the formula is satisfiable somewhere in the interior of a model over $[0, 1]$. To define this relativization we need to use a new atom to indicate points in the interior. Hence the next few definitions.
DEFINITION 25 Given $\phi$ and an atom $q$ which does not appear in $\phi$, we define a map $* = *^\phi_q$ on formulas in $\text{Cl}\phi$ recursively:

1. $*p = p \land q$,
2. $*-\alpha = -(*\alpha) \land q$,
3. $*(\alpha \land \beta) = *(\alpha) \land *(\beta) \land q$,
4. $*U(\alpha, \beta) = U(*\alpha, *\beta) \land q$, and
5. $*S(\alpha, \beta) = S(*\alpha, *\beta) \land q$.

So $*^\phi_q(\phi)$ will be a formula using only $q$ and atoms from $\phi$.

LEMMA 26 $*^\phi_q(\phi)$ is at most 3 times as long as $\phi$.

With the relativization machinery we can then define a relativized mosaic to be one which could correspond to the whole of a $[0, 1]$ structure in which $q$ is true of exactly the interior $]0, 1[$ and the interior is a model of $\phi$.

DEFINITION 27 We say that a $*^\phi_q(\phi)$-mosaic $(A, B, C)$ is $(\phi, q)$-relativized iff

1. $-q$ is in $A$ and no $S(\alpha, \beta)$ is in $A$;
2. $q \in B$ and $-*^\phi_q(\phi) \notin B$; and
3. $-q \in C$ and no $U(\alpha, \beta)$ is in $C$.

Here we confirm that $\phi$ is satisfiable over the reals exactly when we can find such a relativized mosaic. (Lemma from [Rey10]).

LEMMA 28 Suppose that $\phi$ is a formula of $L(U, S)$ and $q$ is an atom not appearing in $\phi$. Then $\phi$ is $\mathbb{R}$-satisfiable iff there is some fully $[0, 1]$-satisfiable $(\phi, q)$-relativized $*^\phi_q(\phi)$-mosaic.

Our satisfiability procedure in [Rey10] was to guess a relativized mosaic $(A, B, C)$ and then check that $(A, B, C)$ is fully $[0, 1]$-satisfiable. Thus we now turn to the question of deciding whether a relativized mosaic is satisfiable.

5 Real Mosaic Systems

In this section we define a concept of a collection or system of mosaics in which each member is decomposable in terms of simpler members. We will later show that being in such a system is (roughly) equivalent to satisfiability. First two of the simpler tactics for decomposition.
5.1 Tactics Lead and Trail

The mirror image tactics lead and trail allow mosaics which can be fully decomposed in terms of themselves along with some other mosaics. In a game setting this is a legitimate way for the game to be won: the player who has to keep providing full decompositions can keep supplying a full decomposition $\langle m \rangle \wedge \sigma$ for $m$ if the other player keeps choosing $m$ to be decomposed. The tactic trail corresponds to an operation in [LL66] for building a new linear order from a simpler one by laying $\omega$ copies of it one after the other towards the future. The tactic lead corresponds to laying the copies towards the past.

**DEFINITION 29** Suppose $\phi \in L(U,S)$, $m$ is a $\phi$-mosaic and $\sigma$ is a non-empty sequence of $\phi$-mosaics. Then, we say that $m$ is fully decomposed by the tactic lead($\sigma$) iff $\langle m \rangle \wedge \sigma$ is a full decomposition of $m$. We say that $m$ is fully decomposed by the tactic trail($\sigma$) iff $\sigma \wedge \langle m \rangle$ is a full decomposition of $m$.

5.2 Shuffles

The term shuffle has been used in the literature (see, for example, [LL66], [BG85] or [Ros82]) to refer to certain methods of constructing a linear structure (often a monadic one) from a thorough mixture of smaller linear structures.

The intention here is similar except we need to deal with mosaics corresponding to linear structures instead of structures themselves. We consider (a mosaic corresponding to) a shuffle $S$ of linear structures $U_0, U_1, \ldots, U_s$, $V_1, V_2, \ldots, V_r$ where each $U_i$ is a singleton structure and each $V_i$ is a non-singleton structure consisting of the concatenation of a finite sequence of other structures. Thus, we actually only consider an MPC set $P_i$ instead of $U_i$ and a non-empty composing sequence $\lambda_i$ of mosaics instead of $V_i$. In this case it is possible to construct a certain set of mosaics such that one, $o$, corresponds to $S$ and each one in the set has a full decomposition in terms of others in the set and/or the mosaics which decompose each $\lambda_i$.

In [Rey10], in this vein, there is a rather complex definition of when a mosaic $o$ is fully decomposed by the tactic shuffle ($\langle P_0, \ldots, P_s \rangle$, $\langle \lambda_1, \ldots, \lambda_r \rangle$). So here is Definition 31 of that paper.

**DEFINITION 30** Suppose $0 \leq r$, each $\lambda_i$ ($1 \leq i \leq r$) is a non-empty composing sequence of $\phi$-mosaics, and $P_0, \ldots, P_s$ ($0 \leq s$) are maximally propositionally consistent subsets of $Cl\phi$.

Suppose $\phi$-mosaic $o = (A, B, C)$ and:
$m' = (A, B, P_0);$  
$y_i = (P_i, B, P_{i+1}) \ (0 \leq i \leq s - 1);$  
$y_s = (P_s, B, P_0);$  
$m'' = (P_0, B, C);$ and  
$\mu = \langle y_0, ..., y_s \rangle.$  

If $r = 0$ suppose $\lambda = \langle \rangle$, the empty sequence, but otherwise, if $r > 0$, suppose:  
$A_i$ is the start of the first mosaic in $\lambda_i (1 \leq i \leq r);$  
$C_i$ is the end of the last mosaic in $\lambda_i (1 \leq i \leq r);$  
$x_0 = (P_0, B, A_1);$  
$x_i = (C_i, B, A_{i+1}), \ (1 \leq i \leq r - 1);$  
$x_r = (C_r, B, P_0);$  
$\lambda = \langle x_0 \rangle \ ^{\lambda_1 \langle \lambda_1 \rangle} \ ^{\lambda_2 \langle \lambda_2 \rangle} \ ^{\lambda_3 \langle \lambda_3 \rangle} \ ^{\lambda_r \langle \lambda_r \rangle} \langle \rangle.$  

Further suppose that $m'$, $m''$, and each $y_i$ and $x_i$ are mosaics.  
Then, we say that $\phi$ is fully decomposed by the tactic shuffle $((P_0, ..., P_s), \langle \lambda_1, ..., \lambda_r \rangle)$ iff the following conditions all hold:  

F1. $\phi$ is fully decomposed by $\langle m' \rangle \ ^{\lambda \mu} \langle m'' \rangle$;  
F2. if $r > 0$, $x_0$ is fully decomposed by $\lambda \ ^{\mu} \langle x_0 \rangle$;  
F3. if $0 < i < r, x_i$ is fully decomposed by $\langle x_i \rangle \ ^{\lambda_{i+1} \langle \lambda_{i+1} \rangle} \ ^{\lambda_r \langle \lambda_r \rangle} \ ^{\lambda_r \langle \lambda_r \rangle} \ ^{\lambda_r \langle \lambda_r \rangle} \ ^{\lambda_1 \langle \lambda_1 \rangle} \ ^{\lambda_2 \langle \lambda_2 \rangle} \ ^{\lambda_3 \langle \lambda_3 \rangle} \ ^{\lambda_r \langle \lambda_r \rangle} \langle \rangle$;  
F4. if $r > 0$, $x_r$ is fully decomposed by $\langle x_r \rangle \ ^{\mu \lambda}$;  
F5. if $0 \leq i < s, y_i$ is fully decomposed by $\langle y_i, y_{i+1}, ..., y_s \rangle \ ^{\lambda \mu} \langle y_0, ..., y_i \rangle$;  
F6. $y_s$ is fully decomposed by $\langle y_s \rangle \ ^{\lambda \mu}$.

We will also present a slightly shorter alternative characterisation that also appeared (and was proved equivalent) in that paper. Later in this paper we will need both characterisations and need to know that they are equivalent.

The forward $K(m)$ property is supposed to hold of an MPC set if that set could be the end of the last mosaic in some $\lambda_i$ where mosaic $m$ is fully decomposed by the tactic shuffle $((P_0, ..., P_s), \langle \lambda_1, ..., \lambda_r \rangle)$. This is the set of formulas from $\text{Cl}(\phi)$ true at the end point of one of the structures $V_i$ referred to above.

**DEFINITION 31** (Definition 32 from [Rey10]) Suppose $\phi \in L(U, S)$ and $m$ is a $\phi$-mosaic. We say that a set $Q \subseteq \text{Cl}(\phi)$ satisfies the forward $K(m)$ property iff $Q$ is MPC and for any $U(\alpha, \beta) \in \text{Cl}(\phi)$ we have $U(\alpha, \beta) \in Q$ iff both $\beta \in \text{cover}(m)$ and (at least) one of the following holds:  

K1. $-\alpha \notin \text{cover}(m);$  
K2. $\alpha \in \text{end}(m);$ or  
K3. $\beta \in \text{end}(m)$ and $U(\alpha, \beta) \in \text{end}(m)$.  

The mirror image is the backwards $K(m)$ property.
LEMMA 32 (Lemma 33 from [Rey10]) Suppose \( \phi \in L(U, S) \), \( m = (A, B, C) \) is a \( \phi \)-mosaic, and each \( P_i \subseteq \text{Cl}\phi \) (\( 0 \leq i \leq s \)) and each \( \lambda_i \) (\( 1 \leq i \leq r \)) is a composing non-empty sequence of \( \phi \)-mosaics.

Then, \( m \) is fully decomposed by the tactic shuffle \( (\langle P_0, \ldots, P_s \rangle, \langle \lambda_1, \ldots, \lambda_r \rangle) \) iff the following seven conditions hold:

- \( S_0 \) : \( B \) is a subset of each \( P_i \) and of the start, end and cover of each mosaic in each \( \lambda_i \);
- \( S_1 \) : each \( P_i \) satisfies both the forward and backwards \( K(m) \) property;
- \( S_2 \) : the start of the first mosaic in each \( \lambda_i \) satisfies the backwards \( K(m) \) property;
- \( S_3 \) : the end of the last mosaic in each \( \lambda_i \) satisfies the forwards \( K(m) \) property;
- \( S_4 \) : \( A \) satisfies the forward \( K(m) \) property;
- \( S_5 \) : \( C \) satisfies the backwards \( K(m) \) property;
- \( S_6 \) : if \( \beta \in \text{Cl}\phi \) but \( \neg\beta \notin B \) then either \( \beta \) is contained in some \( P_i \) or \( \beta \) is contained in the start or end of some mosaic in some \( \lambda_i \).

Note that as \( s \geq 0 \) there is at least one \( P_i \) involved in the shuffle. In a general linear order setting we could define a shuffle with no \( P_i \)s (provided that then \( r > 0 \)) but over the reals it turns out to be crucial to require at least one \( P_i \). This is because, as it is not too hard to see, a shuffle of only non-singleton closed intervals of the reals can not be both Dedekind complete and separable (i.e. having a countable dense suborder).

5.3 The levels that make an RMS

Now we define the hierarchy of membership of the system of mosaics which we need. Mosaics at one level of membership will be constructed from ones at lower levels of membership by concatenation or some combination of the tactics we have introduced above. As we build up, we only want to allow a limited use of leads and trails before a shuffle takes us to the next highest level. As we will only allow nesting of trails and/or leads of depth 2 within shuffles we define some intermediate levels between levels \( n \) and \( n+1 \). So, as we will see now, the levels, in increasing order are actually \( 0, 0^+, 1^-, 1^+, 2^-, 2^+, \ldots \).

DEFINITION 33 For \( \phi \in L(U, S) \), suppose \( S \) is a set of \( \phi \)-mosaics and \( n \geq 0 \).

A \( \phi \)-mosaic \( m \in S \) is a level \( n^+ \) member of \( S \) iff \( m \) is the composition of a sequence of mosaics, each of them being either a level \( n \) member of \( S \) or fully decomposed by the tactics lead(\( \sigma \)) or trail(\( \sigma \)) with each mosaic in \( \sigma \) being a level \( n \) member of \( S \).

A \( \phi \)-mosaic \( m \in S \) is a level \( (n+1)^- \) member of \( S \) iff \( m \) is the composition of a sequence of mosaics, each of them being either a level \( n^+ \) member
of $S$ or fully decomposed by the tactics $\text{lead}(\sigma)$ or $\text{trail}(\sigma)$ with each mosaic in $\sigma$ being a level $n^+$ member of $S$.

A $\phi$-mosaic $m \in S$ is a level $n$ member of $S$ iff $m$ is the composition of a sequence of mosaics with each of them being either a level $n^-$ member of $S$ or a mosaic which is fully decomposed by the tactic $\text{shuffle}(<P_0,\ldots,P_s>,<\sigma_1,\ldots,\sigma_r>)$ with each mosaic in each $\sigma_i$ being a level $n^-$ member of $S$.

Note that it is generally possible for mosaics to be level 0 members of some $S$ provided that they are compositions of mosaics which can be fully decomposed by shuffles in which there are no sequences (i.e. $r = 0$). Thus these mosaics will have an interior which is a dense mixture of points where $P_0,\ldots,P_s$ hold. These are the only mosaics which can be level 0 members of any $S$.

Also note that if $m$ is a level $n$ member of $S$ then $m$ is the composition of $\langle m \rangle$ so $m$ is clearly a level $n^+$ member of $S$. Similarly, level $n^+$ implies level $(n + 1)^-$ and level $n^-$ implies level $n$.

Finally, note that the set $S$ in the definition above may not be closed under composition. It is even possible that a mosaic is a member of $S$ at a certain level by virtue of being a composition of other mosaics each of which, although being fully decomposed by tactics involving only members of $S$, is not itself a members of $S$. Later we will see that for our purposes we mostly work with sets $S$ which are closed under composition.

**Definition 34** For $\phi \in L(U,S)$, a real mosaic system (RMS) of $\phi$-mosaics is a set $S$ of $\phi$-mosaics such that, for every $m \in S$, there exists some $n$ such that $m$ is a level $n$ member of $S$. For any $n$, we say that $S$ is a real mosaic system of depth $n$ iff every $m \in S$ is a level $n$ member of $S$.

Now from Theorem 75 in [Rey10]:

**Theorem 35** Suppose $\phi$ is a formula of $L(U,S)$ and $q$ is an atom not appearing in $\phi$. Suppose $\psi = *^\phi_q(\phi)$ has length $N$.

Then the following are equivalent:
1. $\phi$ is $\mathbb{R}$-satisfiable;
2. there is a $(\phi,q)$-relativized $\psi$-mosaic which appears in some RMS.

**6 Tableaux**

In this section we see how the mosaics and RMS machinery can be the basis of a tableau-style decision procedure. We will start with a formula $\phi$ and determine whether $\phi$ is satisfiable in RTL or not.

The tableaux we construct will be roughly tree-shaped, albeit the traditional upside down tree with a root at the top: predecessors and ancestors
above, successors and descendants below. They can be thought of as structures for organising and representing iterative full decompositions in the RMS.

We imagine trees growing downwards from the root. A node may have children immediately below it, every node except the root has a unique parent. Each node itself and its parent and the parent’s parent and the parent’s parent’s parent e.t.c. form the set of ancestors of the node. We will also impose an earlier-later relation between siblings (children of the same parent) on some trees and represent it by left-to-right ordering in diagrams.

Here are the basic definitions.

**DEFINITION 36**

1. A tree here is just a set (of nodes), with a successor relation determining (as its transitive closure) a derived, reflexive, anti-symmetric, transitive, ancestor relation such that the set of ancestors of any node is finite and well-ordered (by the ancestor relation) and there is a unique root with no ancestors (apart from itself).

2. If node \(x\) has a successor \(y\) then we say that \(x\) is the parent of \(y\) (it is unique) and \(y\) is a child of \(x\). Any other child of \(x\) is called a sibling of \(y\). A node with no children will be called a leaf node.

3. The depth of a node with \(n\) ancestors is \(n\).

4. An ordered tree is a tree with finite numbers of children for each node and a left-right relation which totally orders siblings. The left-right relation does not relate non-siblings.

5. A \(\phi\)-mosaic labelled tree is a map from nodes of a tree to \(\phi\)-mosaics.

The idea, as we will see, is that generally the labels of the children of a node form a full decomposition for the label of the node.

**DEFINITION 37** A \((\phi-)\) tableau (for \(\phi\)-mosaic \(m\)) is a \(\phi\)-mosaic labelled ordered tree with (0) root labelled by \(m\); (1) each node having the labels on the children nodes taken in order forming a full decomposition of the label on the node.

**DEFINITION 38** Define a leaf node to be a clone iff it has the same label as one of its other ancestors. Define a complete node of a tableau to be either a non-leaf, or a clone leaf node. Define a successful tableau as one in which all nodes are complete (otherwise the tableau is incomplete).

As an example see the successful \(U(p,q)\)-tableau in Figure 1. The three sets of formulas appearing are: \(A = \{p, q, U(p, q)\}\), \(B = \{\neg p, q, U(p, q)\}\) and \(C = \{p, \neg q, U(p, q)\}\).
DEFINITION 39 Suppose that $\phi$ is a formula of $L(U,S)$ and $q$ is an atom not appearing in $\phi$. Say $\psi = \ast_q^\phi(\phi)$.

A $\psi$-tableau is a tableau for $\phi$ iff the root is labelled by a $(\phi,q)$-relativized $\ast_q^\phi(\phi)$-mosaic.

7 The Reals

The mosaic tableaux of the last section were quite simple and quite general. They are not adequate for the special properties of the reals.

Thus, in this section we define a $\mathbb{R}$-tableau to be a type of mosaic tableau. However, we impose some subtle restrictions on the labelling as we travel around the tree.

They are essentially simple graph-theoretic properties of the labels on the decomposition tree.

First, we specify that in an $\mathbb{R}$-tableau we do not allow tableau nodes with a single child. Mosaics which have singleton sequences of themselves as full decompositions, are possible in general linear time, they are called units, but not allowed in the reals.
7.1 Approval of the labels

Next we need some machinery to enable the other properties to be defined properly. Assume $\psi \in L(U,S)$.

Note that the cover of child labels is a superset of the cover of their parent: covers do not decrease as we get deeper in the tree.

Now suppose that $T$ is a successful tableau of $\psi$-mosaics.

In order to determine whether $T$ is a successful $R$-tableau we will define an iterative process of approving individual mosaics in the tree. We approve the mosaic labels themselves regardless of how many times a particular label appears in the tableau. Once it is approved, it is approved everywhere that it appears.

The simplest criterion for approval is that a parent label can be approved whenever all the labels of its child nodes are approved. There are a couple of other ways to gain approval that we will outline below.

If, after some iterations, the root label in the tableau is approved then the tableau is a successful $R$-tableau.

If at some stage there are no applicable rules to approve any more nodes, and the root mosaic remains unapproved then the tableau has failed to be a $R$-tableau. We can terminate the check.

7.2 Trails and Leads

The following pattern in the tableau corresponds to a lead tactic and allows the mosaic $m$ to be approved. Suppose $m = m_0$ is decomposed as $\langle m_1 \rangle \wedge \sigma_0$ in $T$, i.e. $m$ is the label of a parent node and $\langle m_1 \rangle \wedge \sigma_0$ are the labels of the children in order. Suppose further that for all $i = 1, 2, ..., m_i$ is decomposed as $\langle m_{i+1} \rangle \wedge \sigma_i$ in $T$. Suppose that all the mosaics appearing in each $\sigma_i$ are already approved.

Finally suppose that some $m_i = m$.

Then we can approve $m$. We say that we approve $m$ as a lead. This will be explained below.

Similarly we can approve mosaics as trails if we have a looping sequence of decompositions all ending in the $m_i$.

7.3 Shuffles

The pattern to allow approval of a mosaic as a shuffle is a little bit more complicated to describe and identify. It can involve a set of more than one (as yet) unapproved mosaic labels.

Some extra notation helps. Suppose $u$ and $v$ are mosaics appearing in $T$. We write $u \leadsto v$ iff there is a sequence $u = m_0, m_1, ..., m_n = v$ of mosaics in

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In an earlier report version of this proof we used more complicated machinery involving an induction on the sizes of the cover sets of mosaic labels. This has been superseded.
respective decompositions such that each $m_i$ is fully decomposed (somewhere in $T$) as $\sigma_i \wedge\langle m_{i+1}\rangle \wedge \pi_i$. Because the covers of mosaics in decompositions are supersets of the cover of the parent, if $u \sim v$ then the cover of $v$ is a superset of the cover of $u$.

The conditions for approving a mosaic as a shuffle are SH1-SH6 as set out below. They are a little complicated so we deal with one at a time. Consider the mosaic $m$ appearing as a label in a tableau.

(SH1) $m$ is an unapproved mosaic.

(SH2) Every unapproved descendent of $m$, including $m$ itself, has some descendent which has at least two separate nodes labelled by unapproved mosaics.

(SH3) All descendants of $m$ which are unapproved have the same cover as $m$.

(SH4) is the requirement that every unapproved descendent $u$ of $m$ (including $m$ itself) has a “crisp start”. That is, there is a sequence $u = m_0, m_1, ...$ of unapproved mosaics in respective decompositions as follows. Each $m_i$ is fully decomposed as $\sigma_i \wedge\langle m_{i+1}\rangle \wedge \pi_i$ where each mosaic in $\sigma_i$ is approved already. Suppose $m_i = m_j$ for some $i < j$. Further, suppose that for each $k = i, i + 1, ..., j - 1$, $\sigma_k$ is actually empty. Thus we identified a start of a possible shuffle involving $u$.

Similarly, (SH5), we require the unapproved descendants of $m$ to have crisp ends using the mirror image construction.

The last check (SH6) before we approved $m$ as a shuffle is to find an unapproved descendent $u$ of $m$ such that $u$ has two adjacent children with unapproved labels $v$ and $w$ that further satisfy the following pattern.

We have a sequence $v = m_0, m_1, ...$ of unapproved mosaics in respective decompositions as follows. Each $m_i$ is fully decomposed as $\sigma_i \wedge\langle m_{i+1}\rangle$. Suppose $m_i = m_j$ for some $i < j$.

And the mirror image condition applied to $w$.

This SH6 corresponds to making sure that there is a point structure taking part in the shuffle, a condition which we have seen ensures Dedekind completeness.

If SH1-6 hold then we can be sure that the shuffle is acceptable and we can approve $m$ as a shuffle.

### 7.4 \(\mathbb{R}\)-tableau by approval

This concludes our account of the approval process that defines a successful \(\mathbb{R}\)-tableau.

**DEFINITION 40** A successful tableau is a successful \(\mathbb{R}\)-tableau iff all mosaic labels can be approved according to the iterative process above.

As an example, we find that the tableau in Figure 1 is a successful \(\mathbb{R}\)-tableau. The mosaic \((B, B, B)\) can be approved as a shuffle, then \((A, B, B)\)
and \((B, B, C)\) separately as a lead and trail, then \((A, B, C)\) because its two children are approved.

## 8 Soundness

In [Rey10], we define a concept of realization intended to capture the idea of a mosaic being satisfiable (over the reals) as far as internal information is concerned: i.e. we ignore formulas of the form \(U(\alpha, \beta)\) in the end or \(S(\alpha, \beta)\) in the start. To do so we generalize the idea of the semantic valuation function—the map which maps a time point to the set of formulas true then—to a more general class of functions (realization maps) which only have some of their properties. This is [Rey10], definition 39:

**DEFINITION 41** Suppose that \(x < y\) from \([0, 1]\). We say that \(\phi\)-mosaic \(m\) is realized by the map \(\mu\) on the closed interval \([x, y]\) iff the following conditions all hold:

1. For each \(z \in [x, y]\), \(\mu(z)\) is a maximally propositionally consistent subset of \(\text{Cl}\phi\);
2. Suppose \(z \in [x, y]\). Then \(U(\alpha, \beta) \in \mu(z)\) iff either
   - \(R2.1\), there is \(u\) such that \(z < u \leq y\) and \(\alpha \in \mu(u)\) and for all \(v\), if \(z < v < u\) then \(\beta \in \mu(v)\) or
   - \(R2.2\), \(\beta \in \mu(y)\), \(U(\alpha, \beta) \in \mu(y)\) and for all \(v\), if \(z < v < y\), then \(\beta \in \mu(v)\);
3. The mirror image of \(R2\) for \(S(\alpha, \beta)\);
4. \(\mu(x)\) is the start of \(m\);
5. \(\mu(y)\) is the end of \(m\); and
6. For each \(\beta \in \text{Cl}\phi\), \(\beta\) is in the cover of \(m\) iff for all \(u\), if \(x < u < y\), \(\beta \in \mu(u)\).

A mosaic \(m\) is said to be realized on \([x, y]\) iff there exists a map \(\mu\) such that \(m\) is realized on \([x, y]\) by \(\mu\). Say mosaic \(m\) is realised in \([0, 1]\) iff for all \(x < y\) from \([0, 1]\), there is \(\mu\) such that \(m\) is realised by \(\mu\) on \([x, y]\). Say that \(m\) is realised iff it is realised on \([0, 1]\).

Consider the mosaic corresponding to an interval in a structure in the sense of definition 16. It should be clear that this mosaic is realized by the semantic valuation function for formulas at points within the interval, i.e. the semantic valuation function is a type of realization map.

Some lemmas from [Rey10].

**LEMMA 42 (Lemma 41 from [Rey10])** Suppose \(\psi\) is a \(L(U, S)\) formula, and \(\psi\)-mosaic \(m\) is the composition of \(m'\) and \(m''\) with each of \(m'\) and \(m''\) being realised.

Then \(m\) is realised.
LEMMA 43 (Lemma 42 from [Rey10]) Suppose \( \psi \) is a \( L(U,S) \) formula, \( \psi \)-mosaic \( m \) is fully decomposed by the tactic lead \( \sigma \) (and similarly trail) and each mosaic in \( \sigma \) is realised. Then \( m \) is realised.

LEMMA 44 (Lemma 44 from [Rey10]) Suppose \( \psi \) is a \( L(U,S) \) formula, \( \psi \)-mosaic \( m \) is fully decomposed by the tactic shuffle \( (\langle P_0, ..., P_s \rangle, \langle \lambda_1, ..., \lambda_r \rangle) \) and each mosaic in each \( \lambda_i \) is realised. Then \( m \) is realised.

8.1 Approval Implies Realised

In the next few lemmas we show that approval in a \( \mathbb{R} \)-tableau implies being realised.

LEMMA 45 Suppose \( \psi \) is a \( L(U,S) \) formula and \( \psi \)-mosaic \( m \) is approved in a \( \mathbb{R} \)-tableau because it children are approved. Then \( m \) is realised.

PROOF: By Lemma 42. \( \square \)

LEMMA 46 Suppose \( \psi \) is a \( L(U,S) \) formula and \( \psi \)-mosaic \( m \) is approved in a \( \mathbb{R} \)-tableau as a lead. Then \( m \) is realised.

PROOF: By Lemma 43. \( \square \)

Similarly trail.

8.2 Shuffle

The final possibility is that \( m \) is approved as a shuffle.

LEMMA 47 Suppose \( \psi \) is a \( L(U,S) \) formula and \( \psi \)-mosaic \( m \) is approved in a \( \mathbb{R} \)-tableau as a shuffle. Then \( m \) is realised.

PROOF: (SH1) \( m \) is an unapproved mosaic.

(SH2) Every unapproved descendent of \( m \), including \( m \) itself, has some descendent which has at least two separate nodes labelled by unapproved mosaics.

(SH3) All descendants of \( m \) which are unapproved have the same cover as \( m \).

(SH4) is the requirement that every unapproved descendent \( u \) of \( m \) (including \( m \) itself) has a “crisp start”. That is, there is a sequence \( u = m_0, m_1, ... \) of unapproved mosaics in respective decompositions as follows. Each \( m_i \) is fully decomposed as \( \sigma_i \wedge \langle m_{i+1} \rangle \wedge \pi_i \) where each mosaic in \( \sigma_i \) is approved already.
Suppose $m_i = m_j$ for some $i < j$. Further, suppose that for each $k = i, i + 1, \ldots, j - 1$, $\sigma_k$ is actually empty. Thus we identified a start of a possible shuffle involving $u$.

Similarly, (SH5), we require the unapproved descendants of $m$ to have crisp ends using the mirror image construction.

The last check (SH6) before we approved $m$ as a shuffle is to find an unapproved descendent $u$ of $m$ such that $u$ has two adjacent children with unapproved labels $v$ and $w$ that further satisfy the following pattern.

We have a sequence $v = m_0, m_1, \ldots$ of unapproved mosaics in respective decompositions as follows. Each $m_i$ is fully decomposed as $\sigma_i \wedge \langle m_{i+1} \rangle$. Suppose $m_i = m_j$ for some $i < j$.

And the mirror image condition applied to $w$.

All the above (SH1-6) were checked before we approved $m$.

Now some useful terminology.

For each mosaic, $v$, which is below $m$ but not yet approved find $\text{pre}(v)$ and $\text{post}(v)$ as follows. Let $v_0 = v, v_1, v_2, \ldots$ be the sequence of initial unapproved mosaics in their respective decompositions. Thus, for each $v_i$, the first unapproved mosaic in the decomposition $F(v_i)$ of $v_i$ is $v_i+1$. Let $\sigma_i$ be the (possibly empty) sequence of light mosaics in $F(v_i)$ before $v_i+1$. As there are only a finite number of mosaics, there is some first $i < j$ such that $v_i = v_j$. We put

$$\text{pre}(v) = \sigma_0 \wedge \sigma_1 \wedge \ldots \wedge \sigma_{i-1}$$

If there are no mosaics in that sequence we say that $\text{pre}(v)$ is just the start of $v$, an MCS set.

Post is mirror.

Let $K$ be the set of unapproved mosaics $v$ below $m$.

We claim that this set defines a shuffle as follows. We define a new set $\Sigma$ of mosaics and point-structures, i.e. MCS subsets from $\text{Cl}(\phi)$.

Suppose $w \in K$ and choose a full decomposition $F(w) = \langle v_1, \ldots, v_k \rangle$ of $w$ from the tree with $v_i \in K$, $v_j \in K$ and $v_k \notin K$ for all $i < k < j$, for some $i \neq j$. Say that $\sigma$ is the possibly empty sequence of mosaics $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$. For all such $w, i, j$, we include in $\Sigma$ a mosaic or point-structure corresponding to the composition of $\text{post}(v_i) + \sigma + \text{pre}(v_j)$.

Note that we will specify that this will be a point structure if $\text{post}(v_i)$ is an MCS set being the end of $v_i$, $\text{post}(v_i) + \sigma + \text{pre}(v_j)$ is empty and $\text{pre}(v_j)$ is an MCS set being the start of $v_j$. In that case $\text{post}(v_i)$ and $\text{pre}(v_j)$ are the same point structure. We just put that single point structure in $\Sigma$. 

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Note that by the shuffle restriction SH6 on $\mathbb{R}$-tableaux, there will be at least one such point structure in $\Sigma$.

Also note that by SH2, $pre(m)$ will just be a finite sequence of mosaics. There is no infinitely repeated part. Similarly $post(m)$.

If we look at a decompositions of $m$ that are deep enough below $m$ then we can find one of the form $pre(m) \land \pi \land post(m)$.

Just keep decomposing mosaics at the start and end.

Let $s$ be the composition of $\pi$. By SH3 and SH2, $s$ will have cover the same as $m$.

In fact we will have the following: $start(s)$ is the end of $pre(m)$; $cover(s)$ is the cover of $m$; and $end(s)$ is the start of $post(m)$.

We can also show that $s$ is fully decomposed by the tactic shuffle ($\langle \Sigma_0 \rangle, \langle \Sigma' \rangle$) where $\Sigma_0$ is the sequence of point-structures in $\Sigma$ in any order and $\Sigma'$ is the rest of $\Sigma$ in any order.

To do so we use Lemma 32. Say $m = (A, B, C)$.

S0 holds as each point and each mosaic appears as a descendant of $m$ which has cover $B$.

S1 holds as each element of $\Sigma_0$ is the start of a $B$ cover mosaic by SH5 and SH6.

S2 holds as the first mosaic starts with the end of $B$ mosaic.

Ditto S3.

S4 holds as $A$ starts the mosaic which starts the shuffle.

Similarly S5.

S6 holds as each $B$ mosaic gets fully decomposed and therefore there is a witness to each such $\beta$ in one of the sequences that we put together to get $\Sigma$.

Recall that $m$ is just $pre(m) + s + post(m)$ and so is realised as well.

$\square$

8.3 Putting it all together

Now the main new lemma.

**LEMMA 48** Suppose $\psi$ is a $L(U, S)$ formula and $\psi$-mosaic $m$ is approved in a $\mathbb{R}$-tableau. Then $m$ is realised.

**PROOF:** We show by induction on the order of approving mosaic labels in a successful $\mathbb{R}$-tableau that all such mosaics are realised. Suppose that all mosaics so far approved are realised. Now suppose that $\psi$-mosaic $m$ appears in a successful tableau $T$ and gets approved.

There are four ways that $m$ can get approved and we consider them case by case.
The simplest way that \( m \) is approved is when it labels a node and all the children nodes are approved. In this case we know that \( m \) is the composition of the child mosaic labels and all of those are realisable. Then \( m \) is approved by Lemma 45.

Another possibility is that \( m \) is approved as a lead. Use lemma 46 and we are done.

Similarly trail and shuffle (Lemma 47). □

**Lemma 49**
Suppose \( \phi \) is a \( L(U,S) \) formula. Say there is an atom \( q \) not appearing in \( \phi \) and \( \psi = \ast_q^\phi(\phi) \) and \( \psi \)-mosaic \( m \) that is \((\phi,q)\)-relativized and has a successful tableau. It is the root of the tableau.

Then \( m \) is satisfied in a structure on the whole of \([0,1]\).

**Proof:** By Lemma 48, as \( m \) appears in a successful tableau then there is \( \mu \) such that \( m \) is realised by \( \mu \) on \([0,1]\).

As \( m \) is \((\phi,q)\)-relativized, \( m \) is satisfied in a structure on the whole of \([0,1]\). □

**Lemma 50**
If \( L(U,S) \) formula \( \phi \) has a successful \( \mathbb{R} \)-tableau then \( \phi \) is \( \mathbb{R} \)-satisfiable.

**Proof:** Suppose \( L(U,S) \) formula \( \phi \) has a successful tableau.

Then there is an atom \( q \) not appearing in \( \phi \) and \( \psi = \ast_q^\phi(\phi) \) and \( \psi \)-mosaic \( m \) that is \((\phi,q)\)-relativized and has a successful tableau. It is the root of the tableau.

By Lemma 49, \( m \) is satisfied in a structure on the whole of \([0,1]\).

By Lemma 28, \( m, \phi \) has a \( \mathbb{R} \)-flowed model. □

9 Completeness

Showing that satisfiable formulas have successful tableaux is not too hard when we can use the levels of an RMS and the way that we can use leads, trails and shuffles to get to the next level. In [Rey10] it was quite clear that these operations correspond to simple repetitive patterns in a decomposition tree. They translate directly to good behaviour in tableaux.

For example, if \( m \) is fully decomposed by tactic lead applied to the sequence \( \sigma \) of mosaics at lower levels, then \( m \) has a tableau starting with a root with children \( m \) and then the mosaics in \( \sigma \) in order. There will be no central sticks as because of the way leads and trails are defined. An induction takes care of the lower level \( \sigma \) mosaics and we are done.

**Lemma 51**
Suppose \( \psi \) is a formula of \( L(U,S) \), \( \psi \)-mosaic \( m \) is fully decomposed by the tactic lead \( \sigma \) (or trail) and each mosaic in \( \sigma \) has a successful \( \mathbb{R} \)-tableau in which \( m \) does not appear.

Then \( m \) has a successful \( \mathbb{R} \)-tableau.
Equally, a shuffle tells us about a set of mutual decompositions which end up leaving a tableau with only lower level mosaics. See Definition 31, page 16/17 of [Rey10]. The \( \mathcal{R} \)-tableau conditions can be checked directly on these decompositions.

**Lemma 52** Suppose \( \psi \) is a formula of \( L(U,S) \), \( \psi \)-mosaic \( m \) is fully decomposed by the tactic shuffle \( \langle P_0, ..., P_s \rangle, \langle \lambda_1, ..., \lambda_r \rangle \) and each mosaic in each \( \lambda_i \) has a successful \( \mathcal{R} \)-tableau in which \( m \) does not appear. Then \( m \) has a successful \( \mathcal{R} \)-tableau.

Put these two lemmas together in an induction and we get:

**Lemma 53** Suppose \( \psi \) is a formula of \( L(U,S) \) and \( \psi \)-mosaic \( m \) appears in an RMS. Then \( m \) is the root of a successful \( \mathcal{R} \)-tableau.

Then use the relativisation results to translate from mosaics to formulas:

**Lemma 54** If \( L(U,S) \) formula \( \phi \) is \( \mathcal{R} \)-satisfiable then \( \phi \) has a successful \( \mathcal{R} \)-tableau.

10 Tableau Correctness Summary

Then we can put the soundness and completeness lemmas together and get our desired overall theorem.

**Theorem 55** \( L(U,S) \) formula \( \phi \) is \( \mathcal{R} \)-satisfiable iff \( \phi \) has a successful \( \mathcal{R} \)-tableau.

11 Termination, Complexity and Implementation Issues

It is easy to see that because we can, without loss of generality, stop at clone nodes, and limit branching factors, only a finite number of different tableaux need be considered for a formula. However, that is the end of the good news. There is an exponential bound on the number of different mosaics for a formula (in terms of its length). This also bounds the length of branches in a tableau. With a linear bound on the branching factor (lemmas in [Rey10]) we thus have a double exponential bound on the size of any tableau in terms of number of nodes. There is thus a triple exponential bound on the number of tableaux which would govern the complexity of any exhaustive search through the tableaux.

However, by guessing a tableau of double exponential size we have a decision procedure that runs in 2-NEXPTIME.
LEMMA 56  In terms of the length of the input formula $\phi$, there is a finite triple exponential bound on the number of tableaux for $\phi$. A decision procedure runs in 2-NEXPTIME.

The complexity of reasoning using such tableaux is thus 2-NEXPTIME. In future work (joint with others) we will report on the possibilities for implementation of this technique. Early Java implementations [Rey11] of a mosaic tableau for US/LIN show that any direct implementation of this tableau technique is quickly overwhelmed by the multi-exponential blow-up in data structures. The number of mosaics for a formula is a particular problem if they all need to be generated and checked. For example, the formula $\phi = U(p, q)$ of length 3 has 2,304 different mosaics to consider; $U(U(p, q), q)$ of length 5 has 22,848 different mosaics; and $U(U(p, q), q) \land \neg U(p, q)$ of length 11 has 228,864 mosaics. Clearly, more intelligent techniques are needed to make practical use of this basic framework.

Note that an implementation of the tableau reasoner for RTL would need two parts. First there is the tableau of mosaic decompositions which has a similar task to that of the USLIN tableau in [Rey11]. The second part is a much less computationally complex check through the successful tableau for the graph restrictions.

12 Conclusion and Future Work

References


