# EXISTENCE AND UNIQUENESS IN PHOTOMETRIC STEREO

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**Doctoral Dissertation**

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**June 1991**

# Statement of Originality

I certify that this thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma at any university, and that, to the best of my knowledge and belief, it does not contain any material previously published or written by another person except where due reference is made in the text.

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*Ryszard Kozera*

# Acknowledgements

I should like to thank Associate Professor Michael J. Brooks, my supervisor, for his helpful suggestions, constant encouragement and patience during my stay at Flinders University. Many thanks are also due to Professor Wojciech Chojnacki for valuable and fruitful discussions. For providing an excellent environment for my Ph. D. studies, I should like to thank all members of the School of Information Science and Technology. I am also grateful for the assistance of Mr. Danny Gibbins in generating and preparing figures. Last but not least, the financial assistance of a Flinders University Postgraduate Scholarship is gratefully appreciated.

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# 0. Introduction

A monochrome photograph of a smooth object will typically exhibit brightness variation, or *shading*. Of interest to researchers in computer vision has been the problem of how object shape may be extracted from image shading. This *shape-from-shading problem* has been shown by Horn [10, Sections 10 and 11] to correspond to that of solving a first-order partial differential equation. Specifically, one seeks a function $u$, representing surface depth in the direction of the $z$-axis, satisfying the *image irradiance equation*

$$R(u_x, u_y) = E(x, y) \tag{0.1}$$

over $\Omega$. Here $R$ is a known function (the so-called *reflectance map*) capturing the illumination and surface reflecting conditions, $E$ is an image formed by orthographic projection of light onto a plane parallel to the $xy$-plane, and $\Omega$ is the image domain. The geometry of the situation is depicted in Figure 1. In this formulation, it is implicitly assumed that

- a small surface portion reflects light independently of its position in space. Thus, scene radiance is dependent only on lighting, surface lustre, and surface normal. By implication, light sources are infinitely far away, and internal surface reflections are disallowed.

- image irradiance corresponding to a surface point is equal to scene radiance.

Note that if function $u$ satisfies (0.1), then so too does $u + c$, where $c$ is a constant. In other words, the image of the surface formed by the graph of $u$ is preserved under a depth shift along the $z$-axis. It is therefore reasonable to assume that solutions to (0.1) are identified with classes of functions that satisfy (0.1) and differ by a constant.

An interesting case obtains when the reflectance map is specified so as to correspond to the situation in which a distant point source illuminates a *Lambertian surface*. A small portion of such a surface acts as a perfect diffuser appearing equally bright from all directions. At first, this might seem to imply that Lambertian surfaces cannot exhibit other than constant shading. However, a curved object will, in general, receive illumination that differs in strength across the surface due to surface foreshortening, and it is this that will be responsible for variation in image brightness. If a small portion with normal direction $(u_x, u_y, -1)$ is illuminated by a distant point source of unit power in direction $p = (p_1, p_2, p_3)$, then, according to Lambert's law, the emitted radiance and, in view of aforementioned assumptions, the reflectance map are given by the cosine of the angle between these two directions. Thus, if $E(x, y)$ denotes the corresponding image, the image irradiance equation for the above situation takes the form

$$\frac{p_1 u_x + p_2 u_y - p_3}{\sqrt{p_1^2 + p_2^2 + p_3^2}\sqrt{u_x^2 + u_y^2 + 1}} = E(x, y). \tag{0.2}$$

Since $E(x, y)$ represents the intensity of the reflected light and as such is non-negative, the domain $\Omega$ of this equation consists of those points for which the left-hand side is non-negative. On the other hand, by the Cauchy-Schwartz-Buniakowski inequality, $E(x, y) \leq 1$, and so effectively the inequalities $0 \leq E(x, y) \leq 1$ hold over $\Omega$.

Given $0 \leq E(x, y) \leq 1$, the questions of the existence and uniqueness of solutions to (0.2) arise naturally. Existence corresponds to the problem of whether a given shading pattern with intensity between 0 and 1 is generated by a genuine Lambertian surface. Uniqueness corresponds to that of whether a shading pattern is due to one and only one Lambertian shape. Some progress in the elucidation of these issues has been made under the assumption that a light source is situated overhead. In this case $p = (0, 0, -1)$ and the corresponding image irradiance equation takes the form

$$\frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} = E(x, y).$$

Of course, $E(x, y) > 0$, so letting $\mathcal{E}(x, y) = \big[E(x, y)\big]^{-2} - 1$, one can rewrite the above as the *eikonal equation*

$$u_x^2 + u_y^2 = \mathcal{E}(x, y). \tag{0.3}$$

Blake *et al.* ([1]; see also Horn and Brooks [11, pp. 29-52]), Brooks [2], Brooks *et al.* [4], Bruss ([7]; see also Horn and Brooks [11, pp. 69-87]), Deift and Sylvester [8], Oliensis [18] and Saxberg [20] contributed important uniqueness results for this equation, whereas Brooks *et al.* [3] and Horn *et al.* [14] established an existence result. While all these results are far from being complete, they indicate, however, that uniqueness is rather exceptional and that existence is subject to many constraints.

In contrast with this, the shape of a Lambertian surface turns out to be uniquely determined by a triplet of images obtained by illuminating a given scene from three different light-source directions (*three-source photometric stereo*). As shown by Horn [10, Subsection 10.13] and Woodham ([22]; see also Horn and Brooks [11, pp. 513-532]), the system

$$\begin{aligned}
\frac{p_1 u_x + p_2 u_y - p_3}{\sqrt{p_1^2 + p_2^2 + p_3^2}\sqrt{u_x^2 + u_y^2 + 1}} &= E_1(x, y), \\
\frac{q_1 u_x + q_2 u_y - q_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}\sqrt{u_x^2 + u_y^2 + 1}} &= E_2(x, y), \\
\frac{r_1 u_x + r_2 u_y - r_3}{\sqrt{r_1^2 + r_2^2 + r_3^2}\sqrt{u_x^2 + u_y^2 + 1}} &= E_3(x, y)
\end{aligned} \tag{0.4}$$

can be reduced to a system of the form

$$\begin{aligned}
u_x &= F_1(x, y), \\
u_y &= F_2(x, y),
\end{aligned}$$

where $F_1$ and $F_2$ are explicitly expressible in terms of $E_1$, $E_2$, $E_3$ and $p_1$, $p_2$, $p_3$, $q_1$, $q_2$, $q_3$, $r_1$, $r_2$, $r_3$.

Less well-understood is the case of *two-source photometric stereo* (see Horn [10, Section 10 and Appendix A.2.2], Horn and Ikeuchi [12], Horn *et al.* [13], Woodham [21, 22]). Here, the shape of a Lambertian surface is to be recovered from a pair of image data obtained by illumination from two different light-source directions. One is thus led to consideration of a system of the form

$$\begin{aligned} \frac{p_1 u_x + p_2 u_y - p_3}{\sqrt{p_1^2 + p_2^2 + p_3^2}\sqrt{u_x^2 + u_y^2 + 1}} &= E_1(x, y), \\ \frac{q_1 u_x + q_2 u_y - q_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}\sqrt{u_x^2 + u_y^2 + 1}} &= E_2(x, y). \end{aligned} \tag{0.5}$$

The main purpose of this thesis is to discuss existence and uniqueness of solutions to (0.5). However, prior to this, we consider shape recovery based on image data obtained by illumination from three different light-source directions. A novelty here will be a presentation of a necessary and sufficient condition for the existence of solutions to (0.4). The main result concerning (0.5) will be the one stating that generically this system has, up to a constant, a unique solution. We shall examine thoroughly exceptional pairs $(E_1, E_2)$ for which there is no such uniqueness. We shall also present necessary and sufficient conditions for the existence of solutions to (0.5). The discussion of both (0.4) and (0.5) will be extended by a number of illustrative examples.

The part of thesis bringing the discussion of three-source and two-source photometric stereo has already been accepted for publication and appeared as [15], [16], and [17].

Upon completion of the present work, R. Onn and A. Bruckstein [19] independently published an analysis of two-source photometric stereo that relates to those results herein up to Theorem 2.7 (the interested reader is referred to [19], where independent elegant proofs may be found). The method used by those authors is different from ours, and the analysis performed is less extensive. In particular, unlike the present thesis, [19] neither gives explicit formulae for the shapes recovered, nor elucidates the nature of exceptional cases of non-uniqueness.

The thesis closes with a supplement presenting a short summary of the results concerning existence and uniqueness of solutions to eikonal equation (0.3) contained in joint papers [3], [4], [5], and [6] by Brooks, Chojnacki and the author. It is included because the author of this thesis contributed approximately one third to this work during his Ph. D. candidature. The supplement is of informal character and is of little direct relevance to the analysis of photometric stereo which is the main focus of this thesis.

# 1. Three-light-source shape recovery

We first consider shape recovery based on the three images obtained by illuminating a given object from three different light-source directions. As was shown by Horn [10, Subsection 10.13] and Woodham [22], the uniqueness problem in this case is of purely algebraic character. In this section, we supplement this result by establishing necessary and sufficient conditions for the existence of a unique solution for a given triplet of images. Finally, to illustrate the significance of these conditions, we exhibit two different classes of images for which there are no genuine shapes. The whole analysis will involve no boundary conditions.

Suppose that a Lambertian surface, represented by the graph of a function $u$ of class $C^1$, is illuminated from three linearly independent directions, namely $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, and $r = (r_1, r_2, r_3)$. Let

$$\Omega_1 = \left\{ (x, y) \in \mathbb{R}^2 : E_1(x, y) = \frac{\langle n|p \rangle}{\|n\|\|p\|} \geq 0 \right\},$$

$$\Omega_2 = \left\{ (x, y) \in \mathbb{R}^2 : E_2(x, y) = \frac{\langle n|q \rangle}{\|n\|\|q\|} \geq 0 \right\},$$

$$\Omega_3 = \left\{ (x, y) \in \mathbb{R}^2 : E_3(x, y) = \frac{\langle n|r \rangle}{\|n\|\|r\|} \geq 0 \right\},$$

where $n = (u_x, u_y, -1)$, $\langle \cdot \mid \cdot \rangle$ denotes the standard Cartesian product in $\mathbb{R}^3$, and $\| \cdot \|$ denotes the corresponding norm. Suppose that $\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3$ is a non-empty domain of $\mathbb{R}^2$. Clearly, the corresponding images $E_1$, $E_2$, and $E_3$ are given by

$$E_1(x,y) = \frac{p_1u_x + p_2u_y - p_3}{\sqrt{p_1^2+p_2^2+p_3^2}\sqrt{u_x^2+u_y^2+1}},$$

$$E_2(x,y) = \frac{q_1u_x + q_2u_y - q_3}{\sqrt{q_1^2+q_2^2+q_3^2}\sqrt{u_x^2+u_y^2+1}}, \tag{1.1}$$

$$E_3(x,y) = \frac{r_1u_x + r_2u_y - r_3}{\sqrt{r_1^2+r_2^2+r_3^2}\sqrt{u_x^2+u_y^2+1}}$$

over $\Omega$. Throughout, unless stated otherwise, functions satisfying (1.1) and differing by a constant will be identified. We have the following:

**Theorem 1.1.** *The first derivatives of $u$ can be expressed in terms of $E_1$, $E_2$, $E_3$, $p$, $q$, and $r$ in the following form:*

$$u_x = \frac{(q_2r_3 - q_3r_2)E_1\|p\| + (p_3r_2 - p_2r_3)E_2\|q\| + (p_2q_3 - p_3q_2)E_3\|r\|}{(q_2r_1 - q_1r_2)E_1\|p\| + (p_1r_2 - p_2r_1)E_2\|q\| + (p_2q_1 - p_1q_2)E_3\|r\|},$$

$$\tag{1.2}$$

$$u_y = \frac{(q_3r_1 - q_1r_3)E_1\|p\| + (p_1r_3 - p_3r_1)E_2\|q\| + (p_3q_1 - p_1q_3)E_3\|r\|}{(q_2r_1 - q_1r_2)E_1\|p\| + (p_1r_2 - p_2r_1)E_2\|q\| + (p_2q_1 - p_1q_2)E_3\|r\|}.$$

**Proof.** If we let $\nu$ be the unit normal to the graph of $u$ with components

$$\nu_1 = \frac{u_x}{\sqrt{u_x^2+u_y^2+1}}, \quad \nu_2 = \frac{u_y}{\sqrt{u_x^2+u_y^2+1}}, \quad \text{and} \quad \nu_3 = \frac{-1}{\sqrt{u_x^2+u_y^2+1}},$$

then system (1.1) can be rewritten in the following form

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} p_1\|p\|^{-1} & p_2\|p\|^{-1} & p_3\|p\|^{-1} \\ q_1\|q\|^{-1} & q_2\|q\|^{-1} & q_3\|q\|^{-1} \\ r_1\|r\|^{-1} & r_2\|r\|^{-1} & r_3\|r\|^{-1} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}.$$

In view of the linear independence of $p$, $q$, and $r$, we have

$$\nu_1 = \frac{\begin{vmatrix} E_1 & p_2\|p\|^{-1} & p_3\|p\|^{-1} \\ E_2 & q_2\|q\|^{-1} & q_3\|q\|^{-1} \\ E_3 & r_2\|r\|^{-1} & r_3\|r\|^{-1} \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} E_1\|p\| & p_2 & p_3 \\ E_2\|q\| & q_2 & q_3 \\ E_3\|r\| & r_2 & r_3 \end{vmatrix}}{\det B},$$

$$\nu_2 = \frac{\begin{vmatrix} p_1\|p\|^{-1} & E_1 & p_3\|p\|^{-1} \\ q_1\|q\|^{-1} & E_2 & q_3\|q\|^{-1} \\ r_1\|r\|^{-1} & E_3 & r_3\|r\|^{-1} \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} p_1 & E_1\|p\| & p_3 \\ q_1 & E_2\|q\| & q_3 \\ r_1 & E_3\|r\| & r_3 \end{vmatrix}}{\det B},$$

$$\nu_3 = \frac{\begin{vmatrix} p_1\|p\|^{-1} & p_2\|p\|^{-1} & E_1 \\ q_1\|q\|^{-1} & q_2\|q\|^{-1} & E_2 \\ r_1\|r\|^{-1} & r_2\|r\|^{-1} & E_3 \end{vmatrix}}{\det A} = \frac{\begin{vmatrix} p_1 & p_2 & E_1\|p\| \\ q_1 & q_2 & E_2\|q\| \\ r_1 & r_2 & E_3\|r\| \end{vmatrix}}{\det B},$$

where

$$A = \begin{pmatrix} p_1\|p\|^{-1} & p_2\|p\|^{-1} & p_3\|p\|^{-1} \\ q_1\|q\|^{-1} & q_2\|q\|^{-1} & q_3\|q\|^{-1} \\ r_1\|r\|^{-1} & r_2\|r\|^{-1} & r_3\|r\|^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}.$$

More explicitly

$$\nu_1 = \frac{(q_2r_3 - q_3r_2)E_1\|p\| + (p_3r_2 - p_2r_3)E_2\|q\| + (p_2q_3 - p_3q_2)E_3\|r\|}{\det B},$$

$$\nu_2 = \frac{(q_3r_1 - q_1r_3)E_1\|p\| + (p_1r_3 - p_3r_1)E_2\|q\| + (p_3q_1 - p_1q_3)E_3\|r\|}{\det B},$$

$$\nu_3 = \frac{(q_1 r_2 - q_2 r_1)E_1\|p\| + (p_2 r_1 - p_1 r_2)E_2\|q\| + (p_1 q_2 - p_2 q_1)E_3\|r\|}{\det B}.$$

Noting that

$$u_x = -\frac{\nu_1}{\nu_3} \qquad \text{and} \qquad u_y = -\frac{\nu_2}{\nu_3},$$

we get (1.2). □

As an immediate consequence, we obtain the following:

**Corollary 1.2.** *With $u$, $E_1$, $E_2$, and $E_3$ as above, if $v$ is a function of class $C^1$ such that* (1.1) *holds with $u$ replaced by $v$ over a domain $\Omega$, then $u = v + \text{const}$.*

**Proof.** It follows from Theorem 1.1 that $u_x = v_x$ and $u_y = v_y$. Since $\Omega$ is connected, $u$ and $v$ can only differ by a constant. □

Notice that Corollary 1.2 establishes the essential uniqueness of the recovery of a Lambertian surface from images generated by three different light sources.

Now we shall derive a necessary and sufficient condition for three given functions $E_1$, $E_2$, and $E_3$ to be interpreted as images of a Lambertian surface illuminated from three given linearly independent directions.

**Theorem 1.3.** *Suppose that $\Omega$ is a simply connected domain in $\mathbb{R}^2$, $E_1$, $E_2$, and $E_3$ are functions of class $C^1$ on $\Omega$ with values in $[0,1]$, and $p$, $q$, $r$ are three linearly independent vectors. In order that there exist a solution $u$ of class $C^2$ to* (1.1)*, it is necessary and sufficient that*

$$\frac{\partial}{\partial y}\left(\frac{(q_2r_3-q_3r_2)E_1\|p\|+(p_3r_2-p_2r_3)E_2\|q\|+(p_2q_3-p_3q_2)E_3\|r\|}{(q_2r_1-q_1r_2)E_1\|p\|+(p_1r_2-p_2r_1)E_2\|q\|+(p_2q_1-p_1q_2)E_3\|r\|}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{(q_3r_1-q_1r_3)E_1\|p\|+(p_1r_3-p_3r_1)E_2\|q\|+(p_3q_1-p_1q_3)E_3\|r\|}{(q_2r_1-q_1r_2)E_1\|p\|+(p_1r_2-p_2r_1)E_2\|q\|+(p_2q_1-p_1q_2)E_3\|r\|}\right). \tag{1.3}$$

*If this condition is fulfilled, then $u$ satisfies* (1.2).

**Proof.** By an elementary result from calculus, equation (1.3) is a necessary and sufficient condition for the existence of a function $u$ of class $C^2$ satisfying (1.2). Now a direct calculation shows that if $u$ satisfies (1.2), then it also satisfies (1.1). □

For a given vector $p$, it is not straightforward to generate an example of a smooth function $0 \le E \le 1$ for which there is no solution to a one-image irradiance equation

$$\frac{p_1u_x+p_2u_y-p_3}{\sqrt{p_1^2+p_2^2+p_3^2}\sqrt{u_x^2+u_y^2+1}}=E(x,y)$$

(see [3] and [14]). However, as our next example shows, it is much easier to find functions $0 \le E_1 \le 1$, $0 \le E_2 \le 1$, $0 \le E_3 \le 1$, and vectors $p$, $q$, $r$ for which there is no solution to (1.1).

**Example 1.4.** Let $p=(1,1,-1)$, $q=(-1,-1,-1)$, and $r=(-1,1,-1)$, and let $E_1(x,y)=(1+x^2+y^2)/2$, $E_2(x,y)=(1-x^2-y^2)/2$, and $E_3(x,y)=x^2/2$. Clearly, $0 \le E_i \le 1$ $(i=1,2,3)$ over $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. The corresponding system (1.1) over $\Omega$ reads

$$\frac{u_x+u_y+1}{\sqrt{3}\sqrt{u_x^2+u_y^2+1}}=\frac{1+x^2+y^2}{2},$$

$$\frac{-u_x - u_y + 1}{\sqrt{3}\sqrt{u_x^2 + u_y^2 + 1}} = \frac{1 - x^2 - y^2}{2}, \tag{1.4}$$

$$\frac{-u_x + u_y + 1}{\sqrt{3}\sqrt{u_x^2 + u_y^2 + 1}} = \frac{x^2}{2}.$$

Since

$$\frac{\partial}{\partial y}\left(\frac{E_1 - E_3}{E_1 + E_2}\right) = y \neq 2x = \frac{\partial}{\partial x}\left(\frac{E_3 - E_2}{E_1 + E_2}\right)$$

over $\Omega \setminus \{(x, y) \in \mathbb{R}^2 : 2x = y\}$, we see that condition (1.3) is not satisfied. Hence, by Theorem 1.3, there is no solution of class $C^2$ to (1.4) over $\Omega$.

Suppose now that a Lambertian surface, represented by the graph of a function $u$ of class $C^1$, is illuminated from three linearly dependent directions $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, and $r = (r_1, r_2, r_3)$, with $p$ and $q$ being linearly independent. Let $\alpha$ and $\beta$ be real numbers, such that

$$\|r\|^{-1}r = \alpha\|p\|^{-1}p + \beta\|q\|^{-1}q \tag{1.5}$$

Let $\Omega_1$, $\Omega_2$, $\Omega_3$, and $\Omega$ be defined as at the beginning of this section. We have the following:

**Proposition 1.5.** *With the assumptions as above, we have that*

$$E_3(x, y) = \alpha E_1(x, y) + \beta E_2(x, y) \tag{1.6}$$

*over* $\Omega$.

**Proof.** Obviously, by (1.1) and (1.5),

$$\begin{aligned} E_3 - \alpha E_1 - \beta E_2 = {} & \frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} \big[u_x(r_1\|r\|^{-1} - \alpha p_1\|p\|^{-1} - \beta q_1\|q\|^{-1}) \\ & + u_y(r_2\|r\|^{-1} - \alpha p_2\|p\|^{-1} - \beta q_2\|q\|^{-1}) \\ & - r_3\|r\|^{-1} + \alpha p_3\|p\|^{-1} + \beta q_3\|q\|^{-1}\big] \\ & = 0. \quad \square \end{aligned}$$

The above proposition can be used to generate an easy example of impossible images.

**Example 1.6.** Let $p = (1,0,0)$, $q = (0,0,-1)$, $r = (1,0,-1)$, and $E_1(x,y) = (1+x^2+y^2)/2$, $E_2(x,y) = (1-x^2-y^2)/2$, and $E_3(x,y) = x^2/2$. Then (1.5) holds with $\alpha = \beta = 1/\sqrt{2}$. Of course, (1.6) fails for otherwise $E_3 = \alpha E_1 + \beta E_2$ implies $x^2 - \sqrt{2} = 0$ over $\Omega = \{(x,y) : x^2 + y^2 < 1\}$, a contradiction.

Observe that if (1.6) holds, then (1.1) reduces to the system

$$\frac{p_1 u_x + p_2 u_y - p_3}{\sqrt{p_1^2 + p_2^2 + p_3^2}\sqrt{u_x^2 + u_y^2 + 1}} = E_1(x,y),$$

$$\frac{q_1 u_x + q_2 u_y - q_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}\sqrt{u_x^2 + u_y^2 + 1}} = E_2(x,y).$$

The analysis of this case will be performed in the next section.

# 2. Existence and uniqueness for two-source photometric stereo

In this chapter, we lay theoretical foundations for the method of two-source photometric stereo. This method consists in recovering the shape of a given object from a pair of image data obtained by illuminating the object from two different light-source directions. We show that generically the shape of a smooth Lambertian surface is uniquely determined by a pair of image data. We examine thoroughly exceptional pairs of image data for which there is no such uniqueness. In addition, we present necessary and sufficient conditions for a given pair of data to be generated by a genuine shape. The entire discussion will make no appeal to any boundary conditions.

Suppose that a Lambertian surface $S$, represented by the graph of a function $u$ of class $C^1$, is illuminated from two linearly independent directions, namely $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$. Let

$$\Omega_1 = \Big\{(x, y) \in \mathbb{R}^2 : E_1(x, y) = \frac{\langle n|p\rangle}{\|n\|\|p\|} \geq 0\Big\},$$

$$\Omega_2 = \Big\{(x, y) \in \mathbb{R}^2 : E_2(x, y) = \frac{\langle n|q\rangle}{\|n\|\|q\|} \geq 0\Big\},$$

where, of course, $n = (u_x, u_y, -1)$. Suppose that $\Omega = \Omega_1 \cap \Omega_2$ is a non-empty domain of $\mathbb{R}^2$. Clearly, the corresponding images $E_1$ and $E_2$ are given by

$$
\begin{aligned}
E_1(x,y) &= \frac{p_1 u_x + p_2 u_y - p_3}{\sqrt{p_1^2+p_2^2+p_3^2}\sqrt{u_x^2+u_y^2+1}}, \\
E_2(x,y) &= \frac{q_1 u_x + q_2 u_y - q_3}{\sqrt{q_1^2+q_2^2+q_3^2}\sqrt{u_x^2+u_y^2+1}}
\end{aligned} \tag{2.1}
$$

over $\Omega$. As in the preceding chapter, unless stated otherwise we shall identify functions that satisfy (2.1) and differ by a constant. We have the following:

**Theorem 2.1.** *The first derivatives of $u$ can be expressed in terms of $E_1$, $E_2$, $p$, and $q$ in the following form:*

$$
\begin{aligned}
u_x &= \frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2 + (p_3q_2 - p_2q_3)\varepsilon\sqrt{\Lambda}}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\varepsilon\sqrt{\Lambda}}, \\
u_y &= \frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2 + (p_1q_3 - p_3q_1)\varepsilon\sqrt{\Lambda}}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\varepsilon\sqrt{\Lambda}},
\end{aligned} \tag{2.2}
$$

*where*

$$
\begin{aligned}
\Lambda &= \Lambda(x,y) \\
&= \|p\|^2\|q\|^2\left[1 - E_1^2(x,y) - E_2^2(x,y)\right] - \langle p|q\rangle\left[\langle p|q\rangle - 2\|p\|\|q\|E_1(x,y)E_2(x,y)\right],
\end{aligned} \tag{2.3}
$$

*and $\varepsilon = \varepsilon(x,y)$ is a function taking values $\pm 1$ in such a way that the function*

$$f(x,y) = \varepsilon(x,y)\sqrt{\Lambda(x,y)}$$

*is continuous.*

**Proof.** Let $\widetilde{e}$

$$\widetilde{e} = \|p\|^{-1}p = \frac{(p_1,p_2,p_3)}{\sqrt{p_1^2+p_2^2+p_3^2}}$$

and let

$$f = q - \langle q|\widetilde{e}\rangle\widetilde{e} = \frac{1}{\|p\|^2}\left(\|p\|^2 q_1 - \langle p|q\rangle p_1,\ \|p\|^2 q_2 - \langle p|q\rangle p_2,\ \|p\|^2 q_3 - \langle p|q\rangle p_3\right).$$

Then

$$\|f\| = \|p\|^{-2}\sqrt{\sum_{1\le n\le 3}(\|p\|^2 q_n - \langle p|q\rangle p_n^2)^2}$$

$$= \|p\|^{-2}\sqrt{\|p\|^4\|q\|^2 + \langle p|q\rangle^2\|p\|^2 - 2\|p\|^2\langle p|q\rangle(p_1q_1 + p_2q_2 + p_3q_3)}$$

$$= \|p\|^{-1}\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}.$$

Note that by the Cauchy-Schwartz-Buniakowski inequality and the linear independence $p$ and $q$, we have $\|p\|^2\|q\|^2 - \langle p|q\rangle^2 > 0$. Put

$$\widetilde{f} = \|f\|^{-1}f = \frac{1}{\|p\|\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}}$$

$$\times\,(\|p\|^2 q_1 - \langle p|q\rangle p_1,\ \|p\|^2 q_2 - \langle p|q\rangle p_2,\ \|p\|^2 q_3 - \langle p|q\rangle p_3).$$

Obviously $\|\widetilde{e}\| = \|\widetilde{f}\| = 1$. Moreover

$$\begin{aligned}
\langle \widetilde{e}|\widetilde{f}\rangle &= \|p\|^{-1}\|f\|^{-1}\langle p|f\rangle \\
&= \|p\|^{-1}\|f\|^{-1}\sum_{1\le n\le 3}(\|p\|^2 p_n q_n - \langle p|q\rangle p_n^2) \\
&= \|p\|^{-1}\|f\|^{-1}\left(\sum_{1\le n\le 3}\|p\|^2 p_n q_n - \sum_{1\le n\le 3}\langle p|q\rangle p_n^2\right) \\
&= \|p\|^{-1}\|f\|^{-1}(\|p\|^2\langle p|q\rangle - \|p\|^2\langle p|q\rangle) \\
&= 0.
\end{aligned}$$

Let $\widetilde{g} = \widetilde{e}\times\widetilde{f}$, where $\times$ denotes the cross product in $\mathbb{R}^3$, that is,

$$\widetilde{g} = (\widetilde{e}_2\widetilde{f}_3 - \widetilde{e}_3\widetilde{f}_2,\ \widetilde{e}_3\widetilde{f}_1 - \widetilde{e}_1\widetilde{f}_3,\ \widetilde{e}_1\widetilde{f}_2 - \widetilde{e}_2\widetilde{f}_1).$$

Of course, $\langle \widetilde{e}|\widetilde{g}\rangle = 0$, $\langle \widetilde{f}|\widetilde{g}\rangle = 0$, and $\|\widetilde{g}\| = 1$. Now

$$\widetilde{g} = \frac{1}{\|p\|^2\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}} \begin{pmatrix} p_2(q_3\|p\|^2 - p_3\langle p|q\rangle) - p_3(q_2\|p\|^2 - p_2\langle p|q\rangle) \\ p_3(q_1\|p\|^2 - p_1\langle p|q\rangle) - p_1(q_3\|p\|^2 - p_3\langle p|q\rangle) \\ p_1(q_2\|p\|^2 - p_2\langle p|q\rangle) - p_2(q_1\|p\|^2 - p_1\langle p|q\rangle) \end{pmatrix}$$

$$= \frac{1}{\|p\|^2\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}}$$

$$\times \Big( \|p\|^2(p_2q_3 - p_3q_2),\ \|p\|^2(p_3q_1 - p_1q_3),\ \|p\|^2(p_1q_2 - p_2q_1) \Big)$$

$$= \frac{1}{\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}}\ (p_2q_3 - p_3q_2,\ p_3q_1 - p_1q_3,\ p_1q_2 - p_2q_1).$$

In the orthonormal basis $\widetilde{e},\ \widetilde{f},\ \widetilde{g}$, the unit normal to $S$,

$$\nu = \left( \frac{u_x}{\sqrt{u_x^2 + u_y^2 + 1}}, \frac{u_y}{\sqrt{u_x^2 + u_y^2 + 1}}, \frac{-1}{\sqrt{u_x^2 + u_y^2 + 1}} \right),$$

can be expressed as

$$\nu = \alpha\widetilde{e} + \beta\widetilde{f} + \gamma\widetilde{g}, \tag{2.4}$$

where

$$\alpha = \langle \nu|\widetilde{e}\rangle = \left\langle \nu \middle| \frac{p}{\|p\|} \right\rangle = \|p\|^{-1}\langle \nu|p\rangle = E_1, \tag{2.5}$$

$$\begin{aligned} \beta &= \langle \nu|\widetilde{f}\rangle = \left\langle \nu \middle| \frac{f}{\|f\|} \right\rangle = \|f\|^{-1}\langle \nu|(q - \langle q|\widetilde{e}\rangle\widetilde{e})\rangle \\ &= \|f\|^{-1}(\langle \nu|q\rangle - \langle q|\widetilde{e}\rangle\langle \nu|\widetilde{e}\rangle) = \|f\|^{-1}\left( \|q\|\left\langle \nu \middle| \frac{q}{\|q\|} \right\rangle - \langle q|\widetilde{e}\rangle E_1 \right) \\ &= \|f\|^{-1}(\|q\|E_2 - \langle q|\widetilde{e}\rangle E_1) \\ &= \frac{\|q\|E_2}{\|p\|^{-1}\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}} - \frac{\|p\|^{-1}\langle p|q\rangle E_1}{\|p\|^{-1}\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}} \\ &= \frac{\|p\|\|q\|E_2 - \langle p|q\rangle E_1}{\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}}, \end{aligned} \tag{2.6}$$

$\gamma = \langle \nu | \widetilde{g} \rangle.$

Since $\nu$ has norm 1 and the basis $\widetilde{e}$, $\widetilde{f}$, $\widetilde{g}$ is orthonormal, it follows that

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

and so

$$\gamma = \varepsilon\sqrt{1 - \alpha^2 - \beta^2}, \tag{2.7}$$

for some function $\varepsilon = \varepsilon(x, y)$ taking values $\pm 1$ in such a way that $\gamma$ is a continuous function. Observe that by (2.5) and (2.6), we get

$$\begin{aligned}
1 - \alpha^2 - \beta^2 &= 1 - E_1^2 - \left( \frac{\|p\|\|q\|E_2 - \langle p|q\rangle E_1}{\sqrt{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}} \right)^2 \\
&= \frac{\|p\|^2\|q\|^2 - \langle p|q\rangle^2 - E_1^2(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2} \\
&\quad - \frac{\|p\|^2\|q\|^2E_2^2 + \langle p|q\rangle^2E_1^2 - 2\|p\|\|q\|\langle p|q\rangle E_1E_2}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2} \\
&= \frac{\|p\|^2\|q\|^2(1 - E_1^2 - E_2^2) - \langle p|q\rangle(\langle p|q\rangle - 2\|p\|\|q\|E_1E_2)}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}.
\end{aligned} \tag{2.8}$$

Taking into account (2.4), (2.5), (2.6), (2.7), and (2.8), we have

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \frac{E_1}{\|p\|}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \frac{\|p\|\|q\|E_2 - \langle p|q\rangle E_1}{\|p\|(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}\begin{pmatrix} q_1\|p\|^2 - p_1\langle p|q\rangle \\ q_2\|p\|^2 - p_2\langle p|q\rangle \\ q_3\|p\|^2 - p_3\langle p|q\rangle \end{pmatrix}$$

$$+\,\varepsilon\frac{\sqrt{\|p\|^2\|q\|^2(1 - E_1^2 - E_2^2) - \langle p|q\rangle(\langle p|q\rangle - 2\|p\|\|q\|E_1E_2)}}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}\begin{pmatrix} p_2q_3 - p_3q_2 \\ p_3q_1 - p_1q_3 \\ p_1q_2 - p_2q_1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p_1}{\|p\|}E_1 + \frac{\|p\|\|q\|E_2 - \langle p|q\rangle E_1}{\|p\|(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}(q_1\|p\|^2 - p_1\langle p|q\rangle) + \frac{(p_2q_3 - p_3q_2)\varepsilon\sqrt{\Lambda}}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2} \\ \frac{p_2}{\|p\|}E_1 + \frac{\|p\|\|q\|E_2 - \langle p|q\rangle E_1}{\|p\|(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}(q_2\|p\|^2 - p_2\langle p|q\rangle) + \frac{(p_3q_1 - p_1q_3)\varepsilon\sqrt{\Lambda}}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2} \\ \frac{p_3}{\|p\|}E_1 + \frac{\|p\|\|q\|E_2 - \langle p|q\rangle E_1}{\|p\|(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}(q_3\|p\|^2 - p_3\langle p|q\rangle) + \frac{(p_1q_2 - p_2q_1)\varepsilon\sqrt{\Lambda}}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2} \end{pmatrix}$$

$$= \frac{1}{\|p\|(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}$$

$$\times \begin{pmatrix} p_1(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)E_1 + (q_1\|p\|^2 - p_1\langle p|q\rangle)(\|p\|\|q\|E_2 - \langle p|q\rangle E_1) \\ +(p_2q_3 - p_3q_2)\|p\|\varepsilon\sqrt{\Lambda} \\ p_2(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)E_1 + (q_2\|p\|^2 - p_2\langle p|q\rangle)(\|p\|\|q\|E_2 - \langle p|q\rangle E_1) \\ +(p_3q_1 - p_1q_3)\|p\|\varepsilon\sqrt{\Lambda} \\ p_3(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)E_1 + (q_3\|p\|^2 - p_3\langle p|q\rangle)(\|p\|\|q\|E_2 - \langle p|q\rangle E_1) \\ +(p_1q_2 - p_2q_1)\|p\|\varepsilon\sqrt{\Lambda} \end{pmatrix}$$

$$= \frac{1}{\|p\|(\|p\|^2\|q\|^2 - \langle p|q\rangle^2)}$$

$$\times \begin{pmatrix} p_1\|p\|^2\|q\|^2E_1 - p_1\langle p|q\rangle^2E_1 + q_1\|p\|^3\|q\|E_2 - p_1\|p\|\|q\|\langle p|q\rangle E_2 \\ -q_1\|p\|^2\langle p|q\rangle E_1 + p_1\langle p|q\rangle^2E_1 + (p_2q_3 - p_3q_2)\|p\|\varepsilon\sqrt{\Lambda} \\ p_2\|p\|^2\|q\|^2E_1 - p_2\langle p|q\rangle^2E_1 + q_2\|p\|^3\|q\|E_2 - p_2\|p\|\|q\|\langle p|q\rangle E_2 \\ -q_2\|p\|^2\langle p|q\rangle E_1 + p_2\langle p|q\rangle^2E_1 + (p_3q_1 - p_1q_3)\|p\|\varepsilon\sqrt{\Lambda} \\ p_3\|p\|^2\|q\|^2E_1 - p_3\langle p|q\rangle^2E_1 + q_3\|p\|^3\|q\|E_2 - p_3\|p\|\|q\|\langle p|q\rangle E_2 \\ -q_3\|p\|^2\langle p|q\rangle E_1 + p_3\langle p|q\rangle^2E_1 + (p_1q_2 - p_2q_1)\|p\|\varepsilon\sqrt{\Lambda} \end{pmatrix}$$

$$
= \frac{1}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}
$$

$$
\times \begin{pmatrix} \|p\|(p_1\|q\|^2 - q_1\langle p|q\rangle)E_1 + \|q\|(q_1\|p\|^2 - p_1\langle p|q\rangle)E_2 \\ +(p_2q_3 - p_3q_2)\varepsilon\sqrt{\Lambda} \\ \|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)E_1 + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)E_2 \\ +(p_3q_1 - p_1q_3)\varepsilon\sqrt{\Lambda} \\ \|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 \\ +(p_1q_2 - p_2q_1)\varepsilon\sqrt{\Lambda} \end{pmatrix}
$$

Now (2.2) follows upon noting that $u_x = -\nu_1/\nu_3$ and $u_y = -\nu_2/\nu_3$. □

If $\Lambda \geq 0$, the right-hand sides of (2.2) give candidate partial derivatives of solutions to the system (2.1) in terms of two given images and light-source directions. Let $(v^1, v^2)$ denote one of such candidate pairs. Then the vector field $(v^1, v^2)$ may or may not be the gradient of a function $v$ whose partial derivatives satisfy (2.1). Our analysis of the existence and uniqueness of the solutions to the problem (2.1) will therefore be generically confined to the set of functions of class $C^2$, where the *integrability condition* ($v^1_y = v^2_x$) is a necessary and sufficient condition for the vector field $(v^1, v^2)$ of class $C^1$ to be the gradient of some function $v$ of class $C^2$ over a simply connected domain. In the next subsections we will discuss the problems of existence and uniqueness in two-source photometric stereo by first presenting local results (Subsections 2.1 and 2.2) and then global results (Subsection 2.3).

### 2.1. Case $\Lambda > 0$

When $\Lambda$ is positive over $\Omega$, the function $\varepsilon$ appearing in (2.3) must take one of the values 1 or -1 since gradient $(u_x, u_y)$ has to be a continuous function. As an immediate consequence, we have the following:

**Corollary 2.2.** *Let $E_1$ and $E_2$ be continuous functions on a domain $\Omega$ in $\mathbb{R}^2$ with values in $[0,1]$. Suppose that $\Lambda > 0$ over $\Omega$. Then there exist at most two solutions $u^+$ and $u^-$ of class $C^1$ to (2.1) having first-order derivatives in the form*

$$u_x^\pm = \frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2 \pm (p_3q_2 - p_2q_3)\sqrt{\Lambda}}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda}},$$

$$u_y^\pm = \frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2 \pm (p_1q_3 - p_3q_1)\sqrt{\Lambda}}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda}}. \tag{2.9}$$

**Proof.** The theorem follows immediately upon observing that $\varepsilon(x,y)\sqrt{\Lambda(x,y)}$ being a continuous function implies that either $\varepsilon \equiv 1$ or $\varepsilon \equiv -1$ in (2.2) over $\Omega$. □

By an elementary argument from calculus, we also obtain

**Corollary 2.3.** *Let $E_1$ and $E_2$ be functions of class $C^1$ over a simply connected region $\Omega$ of $\mathbb{R}^2$ with values in $[0,1]$. Suppose that $\Lambda > 0$ on $\Omega$ and that, for each choice of sign,*

$$\sigma^\pm = \|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda}$$

*does not vanish over $\Omega$. Then a necessary and sufficient condition for the existence of exactly two solutions of class $C^2$ to (2.1) is, for each choice of sign,*

$$\frac{\partial}{\partial y}\left(\frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2 \pm (p_3q_2 - p_2q_3)\sqrt{\Lambda}}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda}}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2 \pm (p_1q_3 - p_3q_1)\sqrt{\Lambda}}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda}}\right). \tag{2.10}$$

*If this integrability condition is fulfilled, then these two solutions satisfy* (2.9).

Note that if $E_1$, $E_2$ are derived from a genuine solution, then at least one of the $\sigma^\pm$ does not vanish. It will be shown later that in the case where $p = (0,0,-1)$

and $\|q\| = 1$ if this solution is of class $C^2$ and in addition satisfies a certain second-order linear PDE, namely equation (2.14), then there exists still another solution of class $C^2$ to (2.1). This will imply that both $\sigma^{\pm}$ are non-vanishing. Observe also that

$$u_x^+ = u_x^- \quad \text{and} \quad u_y^+ = u_y^-$$

if

$$p_1q_3 - p_3q_1 = p_3q_2 - p_2q_3 = p_1q_2 - p_2q_1 = 0.$$

The latter identities, however, never hold as vectors $p$ and $q$ are assumed to be linearly independent.

Corollary 2.3 can be illustrated by the following example.

**Example 2.4.** Let $p = (0, 0, -1)$ and $q = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, and let $E_1(x, y) = (x^2 + y^2 + 1)^{-1/2}$ and $E_2(x, y) = (x + y + 1)\big[3(x^2 + y^2 + 1)\big]^{-1/2}$. Consider the corresponding image irradiance equations

$$\begin{aligned} \frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} &= \frac{1}{\sqrt{x^2 + y^2 + 1}}, \\ \frac{u_x + u_y + 1}{\sqrt{3}\sqrt{u_x^2 + u_y^2 + 1}} &= \frac{x + y + 1}{\sqrt{3}\sqrt{x^2 + y^2 + 1}} \end{aligned} \tag{2.11}$$

over $\Sigma = \{(x, y) \in \mathbb{R}^2 : x + y + 1 \geq 0,\ x < y\}$. Clearly $\sigma^{\pm} = (-2/3)(x^2 + y^2 + 1)^{-1/2} \neq 0$ and

$$\Lambda = \frac{(x - y)^2}{3(x^2 + y^2 + 1)}.$$

Notice that $\Lambda$ is positive on $\Sigma$. Now (2.9) takes the form

$$u_x^+ = y, \quad u_y^+ = x \text{ and } u_x^- = x, \quad u_y^- = y,$$

and, for each choice of sign, the integrability condition (2.10),

$$(u_x^{\pm})_y = (u_y^{\pm})_x$$

is obviously satisfied. By Corollary 2.3, there exist exactly two solutions $u^+$ and $u^-$ of class $C^2$ to (2.11) over $\Sigma$. A straightforward calculation shows that

$$u^+(x,y) = xy$$

and

$$u^-(x,y) = \frac{1}{2}(x^2+y^2).$$

Now we shall disentangle the meaning of condition (2.10) for images $E_1$ and $E_2$ generated by a genuine Lambertian surface, that is, when (2.1) is satisfied for a certain function $u$ of class $C^2$. The discussion to follow concerns the case in which one of the two light-source vectors is overhead and both vectors are normalized. These technical assumptions scarcely affect the generality of considerations and are made for the sake of convenience.

In order not to disrupt the course of the proofs of main theorems, we first formulate two lemmata whose proofs will be deferred to Appendices 1 and 2.

**Lemma 2.5.** *Suppose that* $p = (p_1, p_2, p_3)$ *and* $q = (q_1, q_2, q_3)$ *satisfy* $\|p\| = \|q\| = 1$. *Let* $u$ *be a solution to* (2.1). *Then*

$$\Lambda = \frac{\left[u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1) + p_2q_1 - p_1q_2\right]^2}{u_x^2 + u_y^2 + 1}, \tag{2.12}$$

*where*

$$\Lambda(x,y) = 1 - E_1^2(x,y) - E_2^2(x,y) - \langle p|q\rangle\left[\langle p|q\rangle - 2E_1(x,y)E_2(x,y)\right],$$

$$E_1(x,y) = \frac{p_1u_x + p_2u_y - p_3}{\sqrt{u_x^2 + u_y^2 + 1}}, \quad \text{and} \quad E_2(x,y) = \frac{q_1u_x + q_2u_y - q_3}{\sqrt{u_x^2 + u_y^2 + 1}}.$$

Note that the last lemma implies that if a pair $(E_1, E_2)$ is generated from a genuine surface, then $\Lambda$ is non-negative.

**Lemma 2.6.** *Suppose that* $p = (p_1, p_2, p_3)$ *and* $q = (q_1, q_2, q_3)$ *satisfy* $\|p\| = \|q\| = 1$. *Given a function* $u$ *of class* $C^1$ *over a domain* $\Omega$, *let* $E_1$ *and* $E_2$ *be defined by* (2.1). *Suppose that* $\Lambda$, *given by* (2.3), *is positive and, for each choice of sign,*

$$\sigma^{\pm} = (p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 \pm (p_1 q_2 - p_2 q_1)\sqrt{\Lambda}$$

*does not vanish over* $\Omega$. *Let*

$$\theta = u_x(p_2 q_3 - p_3 q_2) - u_y(p_1 q_3 - p_3 q_1) + p_2 q_1 - p_1 q_2.$$

*Then*

$$\frac{(q_1\langle p|q\rangle - p_1)E_1 + (p_1\langle p|q\rangle - q_1)E_2 + (p_3 q_2 - p_2 q_3)\varepsilon\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1 q_2 - p_2 q_1)\varepsilon\sqrt{\Lambda}}$$

$$= \begin{cases} u_x & \text{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta > 0, \\ \dfrac{(a^2 - b^2 - c^2)u_x + 2acu_y + 2ab}{2abu_x + 2bcu_y + b^2 - a^2 - c^2} & \text{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta < 0 \end{cases}$$

*and*

$$\frac{(q_2\langle p|q\rangle - p_2)E_1 + (p_2\langle p|q\rangle - q_2)E_2 + (p_1 q_3 - p_3 q_1)\varepsilon\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1 q_2 - p_2 q_1)\varepsilon\sqrt{\Lambda}}$$

$$= \begin{cases} u_y & \text{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta > 0, \\ \dfrac{2acu_x + (c^2 - a^2 - b^2)u_y + 2bc}{2abu_x + 2bcu_y + b^2 - a^2 - c^2} & \text{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta < 0, \end{cases}$$

*where*

$$a = p_3 q_2 - p_2 q_3, \quad b = p_1 q_2 - p_2 q_1, \quad \text{and } c = p_1 q_3 - p_3 q_1,$$

*and the function* $\varepsilon = \varepsilon(x, y)$ *is everywhere constant taking value* $\pm 1$.

We now prove the following:

**Theorem 2.7.** *Let $p = (0, 0, -1)$ and $q = (q_1, q_2, q_3)$ be such that $q_1^2 + q_2^2 > 0$ and $\|q\| = 1$, and let $u$ be a function of class $C^2$ on a simply connected open subset $\Omega$ of $\mathbb{R}^2$. Suppose that functions $E_1$ and $E_2$ are given by*

$$E_1(x, y) = \frac{1}{\sqrt{u_x^2 + u_y^2 + 1}},$$
$$E_2(x, y) = \frac{q_1 u_x + q_2 u_y - q_3}{\sqrt{u_x^2 + u_y^2 + 1}}. \tag{2.13}$$

*Suppose, moreover, that $\Lambda > 0$ over $\Omega$. In order that there exist a solution of class $C^2$ to (2.13) different from $u$, it is necessary and sufficient that $u$ satisfy*

$$q_1 q_2 \Big( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \Big) + (q_1^2 - q_2^2) \frac{\partial^2 u}{\partial x \partial y} = 0. \tag{2.14}$$

**Proof.** Note that

$$\sigma^{\pm} = \sigma = (q_3^2 - 1) \frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} \neq 0$$

as $\|q\| = 1$ and $q_1^2 + q_2^2 > 0$. By Lemma 2.5

$$\Lambda = \frac{(q_2 u_x - q_1 u_y)^2}{u_x^2 + u_y^2 + 1}.$$

Suppose that

$$\theta = q_2 u_x - q_1 u_y > 0.$$

With $\varepsilon(x, y) \equiv -1$, Lemma 2.6 yields

$$\frac{-q_1 q_3 E_1 - q_1 E_2 + q_2 \sqrt{\Lambda}}{(q_3^2 - 1) E_1} = \frac{(q_2^2 - q_1^2) u_x - 2 q_1 q_2 u_y}{-(q_1^2 + q_2^2)} = \frac{2 q_1 q_2 u_y - (q_2^2 - q_1^2) u_x}{q_1^2 + q_2^2}$$

and

$$\frac{-q_2 q_3 E_1 - q_2 E_2 - q_1 \sqrt{\Lambda}}{(q_3^2 - 1) E_1} = \frac{-2 q_1 q_2 u_x + (q_1^2 - q_2^2) u_y}{-(q_1^2 + q_2^2)} = \frac{2 q_1 q_2 u_x - (q_1^2 - q_2^2) u_y}{q_1^2 + q_2^2}.$$

By Corollary 2.3, a necessary and sufficient condition for the existence of exactly two solutions of class $C^2$ to (2.13) is

$$\frac{\partial}{\partial y}\left(\frac{2q_1q_2u_y - (q_2^2 - q_1^2)u_x}{q_1^2 + q_2^2}\right) = \frac{\partial}{\partial x}\left(\frac{2q_1q_2u_x - (q_1^2 - q_2^2)u_y}{q_1^2 + q_2^2}\right).$$

A simple calculation shows that this condition can be rewritten as

$$q_1q_2\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right) + (q_1^2 - q_2^2)\frac{\partial^2 u}{\partial x \partial y} = 0,$$

as specified in (2.14).

The case $\theta < 0$ is treated analogously. □

The last theorem gives a necessary and sufficient condition for the system (2.13), having at least one solution $u$ of class $C^2$, to admit another solution $v$ of class $C^2$. As equation (2.14) is not generically satisfied by $u$, the uniqueness of this solution to (2.13) in the class of $C^2$ functions is therefore in most cases assured. In other words, the integrability condition disambiguates surface recovery obtained from two image patterns (see Example 2.15). In connection with this theorem, a natural question arises about the shape of solutions to (2.14). This is answered by the following results.

**Theorem 2.8.** *Suppose that* $q_1^2 + q_2^2 > 0$. *Any solution* $u$ *of class* $C^2$ *to* (2.14) *over an open convex* $\Omega$ *is given by*

$$u(x,y) = \begin{cases} \phi(q_1x + q_2y) + \psi(q_1y - q_2x) & \text{if } q_1q_2 \neq 0, \\ \phi(x) + \psi(y) & \text{if } q_1q_2 = 0 \end{cases} \tag{2.15}$$

*for some functions* $\phi$ *and* $\psi$ *of class* $C^2$; *conversely, for any functions* $\phi$ *and* $\psi$ *of class* $C^2$, *the above formula defines a solution of class* $C^2$ *to* (2.14).

**Proof.** It is a matter of simple calculation to verify that if $u$ is given by (2.15) for some functions $\phi$ and $\psi$ of class $C^2$, then $u$ is a solution of class $C^2$ to (2.14).

Suppose now that $u$ is a solution of class $C^2$ to (2.14). Suppose, moreover, that $q_1 q_2 \neq 0$. Let $T$ be a linear transformation of $\mathbb{R}^2$ into itself given by

$$\xi(x,y) = q_1 x + q_2 y,$$
$$\eta(x,y) = -q_2 x + q_1 y.$$

Since

$$\begin{vmatrix} q_1 & q_2 \\ -q_2 & q_1 \end{vmatrix} = q_1^2 + q_2^2 > 0$$

it follows that $T$ is a linear isomorphism. Let

$$\bar{u}(\xi,\eta) = u\Big(T^{-1}(\xi,\eta)\Big).$$

Then, of course,

$$u_x = q_1 \bar{u}_\xi - q_2 \bar{u}_\eta,$$
$$u_y = q_2 \bar{u}_\xi + q_1 \bar{u}_\eta.$$

Moreover

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x}\Big(q_1 \bar{u}_\xi - q_2 \bar{u}_\eta\Big) = q_1(q_1 \bar{u}_{\xi\xi} - q_2 \bar{u}_{\xi\eta}) - q_2(q_1 \bar{u}_{\eta\xi} - q_2 \bar{u}_{\eta\eta}) \\ &= q_1^2 \bar{u}_{\xi\xi} - 2q_1 q_2 \bar{u}_{\xi\eta} + q_2^2 \bar{u}_{\eta\eta}, \end{aligned} \tag{2.16}$$

$$\begin{aligned} u_{yy} &= \frac{\partial}{\partial y}\Big(q_2 \bar{u}_\xi + q_1 \bar{u}_\eta\Big) = q_2(q_2 \bar{u}_{\xi\xi} + q_1 \bar{u}_{\xi\eta}) + q_1(q_2 \bar{u}_{\eta\xi} + q_1 \bar{u}_{\eta\eta}) \\ &= q_2^2 \bar{u}_{\xi\xi} + 2q_1 q_2 \bar{u}_{\xi\eta} + q_1^2 \bar{u}_{\eta\eta}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} u_{xy} &= \frac{\partial}{\partial x}\Big(q_2 \bar{u}_\xi + q_1 \bar{u}_\eta\Big) = q_2(q_1 \bar{u}_{\xi\xi} - q_2 \bar{u}_{\xi\eta}) + q_1(q_1 \bar{u}_{\eta\xi} - q_2 \bar{u}_{\eta\eta}) \\ &= q_1 q_2 \bar{u}_{\xi\xi} + (q_1^2 - q_2^2)\bar{u}_{\xi\eta} - q_1 q_2 \bar{u}_{\eta\eta}. \end{aligned} \tag{2.18}$$

Combining (2.16), (2.17), and (2.18) with (2.14), we get

$$\Big[4q_1^2 q_2^2 + (q_1^2 - q_2^2)^2\Big]\bar{u}_{\xi\eta} = 0$$

or, equivalently,

$$\bar{u}_{\xi\eta} = 0.$$

It is a standard result in PDE's that $\bar{u}$ satisfying the last equation over the convex set $T(\Omega)$ can be expressed in the form

$$\bar{u}(\xi, \eta) = \phi(\xi) + \psi(\eta)$$

for some functions $\phi$ and $\psi$ of class $C^2$. Hence

$$u(x, y) = \bar{u}(\xi, \eta) = \phi(q_1 x + q_2 y) + \psi(q_1 y - q_2 x).$$

If $q_1 q_2 = 0$, then clearly $u$ satisfies

$$u_{xy} = 0,$$

and as such takes the form

$$u(x, y) = \phi(x) + \psi(y)$$

for some functions $\phi$ and $\psi$ of class $C^2$. $\square$

The next result will show how the functions $\phi$ and $\psi$ in the representation (2.15) can be expressed in terms of the initial function $u$.

**Lemma 2.9.** *Suppose that $q_1^2 + q_2^2 > 0$. Let $u$ be a function satisfying (2.15) for some functions $\phi$ and $\psi$ of class $C^2$. Then*

$$\phi(x) = \begin{cases} u\Big(\frac{q_1}{q_1^2 + q_2^2} x, \frac{q_2}{q_1^2 + q_2^2} x\Big) - c & \text{if } q_1 q_2 \neq 0, \\ \\ u(x, 0) - c & \text{if } q_1 q_2 = 0, \end{cases}$$

$$\tag{2.19}$$

$$\psi(x) = \begin{cases} u\Big(\frac{-q_2}{q_1^2 + q_2^2} x, \frac{q_1}{q_1^2 + q_2^2} x\Big) - u(0,0) + c & \text{if } q_1 q_2 \neq 0, \\ \\ u(0, x) - u(0,0) + c & \text{if } q_1 q_2 = 0, \end{cases}$$

*where $c$ is an arbitrary constant.*

**Proof.** Suppose that $q_1q_2 \neq 0$. Then

$$u\Big(\frac{q_1}{q_1^2+q_2^2}x, \frac{q_2}{q_1^2+q_2^2}x\Big) = \phi\Big(\frac{q_1^2}{q_1^2+q_2^2}x + \frac{q_2^2}{q_1^2+q_2^2}x\Big) + \psi\Big(\frac{q_1q_2}{q_1^2+q_2^2}x - \frac{q_1q_2}{q_1^2+q_2^2}x\Big)$$

$$= \phi(x) + \psi(0).$$

So, with $c = \psi(0)$,

$$\phi(x) = u\Big(\frac{q_1}{q_1^2+q_2^2}x, \frac{q_2}{q_1^2+q_2^2}x\Big) - c. \tag{2.20}$$

Analogously

$$u\Big(\frac{-q_2}{q_1^2+q_2^2}x, \frac{q_1}{q_1^2+q_2^2}x\Big) = \phi\Big(\frac{-q_1q_2}{q_1^2+q_2^2}x + \frac{q_1q_2}{q_1^2+q_2^2}x\Big) + \psi\Big(\frac{q_1^2}{q_1^2+q_2^2}x + \frac{q_2^2}{q_1^2+q_2^2}x\Big)$$

$$= \phi(0) + \psi(x).$$

Hence

$$\psi(x) = u\Big(\frac{-q_2}{q_1^2+q_2^2}x, \frac{q_1}{q_1^2+q_2^2}x\Big) - \phi(0).$$

Since, by (2.20),

$$\phi(0) = u(0,0) - c,$$

we have

$$\psi(x) = u\Big(\frac{-q_2}{q_1^2+q_2^2}x, \frac{q_1}{q_1^2+q_2^2}x\Big) - u(0,0) + c.$$

Assume now that $q_1q_2 = 0$. Then

$$u(x,0) = \phi(x) + \psi(0).$$

So, with $c = \psi(0)$,

$$\phi(x) = u(x,0) - c, \tag{2.21}$$

and

$$u(0,x) = \phi(0) + \psi(x).$$

Since, by (2.21),

$$\phi(0) = u(0,0) - c,$$

it follows that

$$\psi(x) = u(0,x) - u(0,0) + c. \quad \square$$

In accordance with Theorem 2.7, if a Lambertian surface corresponding to the graph of a function $u$ generates two images that can be interpreted as images of another Lambertian surface corresponding to the graph of another function $v$, then $u$ satisfies (2.14) and hence, by Theorem 2.8, takes the form

$$u(x,y) = \begin{cases} \phi_1(q_1x + q_2y) + \psi_1(q_1y - q_2x) & \text{if } q_1q_2 \neq 0, \\ \phi_1(x) + \psi_1(y) & \text{if } q_1q_2 = 0 \end{cases}$$

for some functions $\phi_1$ and $\psi_1$ of class $C^2$. Of course, $v$ also obeys (2.14) and so we have

$$v(x,y) = \begin{cases} \phi_2(q_1x + q_2y) + \psi_2(q_1y - q_2x) & \text{if } q_1q_2 \neq 0, \\ \phi_2(x) + \psi_2(y) & \text{if } q_1q_2 = 0 \end{cases}$$

for some functions $\phi_2$ and $\psi_2$ of class $C^2$. A natural question arises about the relationship between the pairs $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$. This is answered by the following:

**Theorem 2.10.** *Let $p = (0,0,-1)$ and $q = (q_1, q_2, q_3)$ be such that $q_1^2 + q_2^2 > 0$ and $\|q\| = 1$. Let $u$ be a function of class $C^2$ defined over an open convex $\Omega$, satisfying*

$$u(x,y) = \begin{cases} \phi(q_1x + q_2y) + \psi(q_1y - q_2x) & \text{if } q_1q_2 \neq 0, \\ \phi(x) + \psi(y) & \text{if } q_1q_2 = 0 \end{cases}$$

*for some functions $\phi$ and $\psi$ of class $C^2$. Let $E_1$ and $E_2$ are given by (2.13). Suppose, moreover, that $\Lambda > 0$ over $\Omega$. Then the other solution $v$ of class $C^2$ to (2.13) can be expressed in the "conjugate" form*

$$v(x,y) = \begin{cases} \phi(q_1x + q_2y) - \psi(q_1y - q_2x) & \text{if } q_1q_2 \neq 0, \\ -\phi(x) + \psi(y) & \text{if } q_1 = 0, \\ \phi(x) - \psi(y) & \text{if } q_2 = 0. \end{cases}$$

**Proof.** Suppose first that $q_1 q_2 \neq 0$. Then

$$u_x = q_1 \phi' - q_2 \psi',$$

$$u_y = q_2 \phi' + q_1 \psi'.$$

By Corollary 2.3 and Lemma 2.6, we get

$$\begin{aligned} v_x &= \frac{2q_1q_2(q_2\phi' + q_1\psi') - (q_2^2 - q_1^2)(q_1\phi' - q_2\psi')}{q_1^2 + q_2^2} \\ &= \frac{2q_1q_2^2 - q_1q_2^2 + q_1^3}{q_1^2 + q_2^2}\phi' + \frac{2q_1^2q_2 - q_1^2q_2 + q_2^3}{q_1^2 + q_2^2}\psi' \\ &= q_1\phi' + q_2\psi'. \end{aligned}$$

Similarly

$$\begin{aligned} v_y &= \frac{2q_1q_2(q_1\phi' - q_2\psi') - (q_1^2 - q_2^2)(q_2\phi' + q_1\psi')}{q_1^2 + q_2^2} \\ &= \frac{2q_1^2q_2 - q_1^2q_2 + q_2^3}{q_1^2 + q_2^2}\phi' + \frac{-2q_1q_2^2 + q_1q_2^2 - q_1^3}{q_1^2 + q_2^2}\psi' \\ &= q_2\phi' - q_1\psi'. \end{aligned}$$

Now it is easy to check that

$$v(x, y) = \phi(q_2 y + q_1 x) - \psi(q_1 y - q_2 x).$$

Suppose now that $q_1 = 0$. Then, by Corollary 2.3 and Lemma 2.6, we obtain

$$v_x = \frac{-q_2^2\phi'}{q_2^2} = -\phi',$$

$$v_y = \frac{q_2^2\psi'}{q_2^2} = \psi',$$

and hence

$$v(x, y) = -\phi(x) + \psi(y).$$

The case $q_2 = 0$ is treated analogously. $\square$

The last results give analytical representations of the two solutions $u$ and $v$ of class $C^2$ to (2.13) and establish a relationship between those representations.

We can now present two alternative ways of finding those two solutions (provided they exist). On the one hand, as indicated in Theorem 2.1, one can recover $u$ and $v$ by direct calculation of gradients $(u_x, u_y)$ and $(v_x, v_y)$ and next applying the formulae

$$u(x,y) = \int_\gamma u_x dx + u_y dy + u(x_0, y_0) \quad \text{and} \quad v(x,y) = \int_\gamma v_x dx + v_y dy + v(x_0, y_0),$$

where $\gamma \subset \Omega$ is a smooth curve joining points $(x, y)$ and $(x_0, y_0)$ in $\Omega$. On the other hand, having found one of those solutions, say $u$, by using, for example, the previous method, and applying Lemma 2.9, we can express functions $\phi$ and $\psi$, introduced in Theorem 2.5, in terms of $u$. Finally, by applying Theorem 2.10, the second solution $v$ can easily be determined in terms of functions $\phi$ and $\psi$ (see Example 2.14).

In the case of overhead, single-point-source illumination, if $u$ is a solution to the shape-from-shading problem, then so too is any $\pm u + c$. In particular, any convex solution to the eikonal equation generates a concave solution whose graph has the same Gaussian curvature as the graph of the convex solution, and *vice versa*. As shown in the next corollary, this is not the case for two-source photometric stereo.

**Corollary 2.11.** *Let $p = (0, 0, -1)$ and $q = (q_1, q_2, q_3)$ be such that $q_1^2 + q_2^2 > 0$ and $\|q\| = 1$. Assume that $\Lambda$ is positive over an open convex $\Omega$. Suppose, moreover, that there exist exactly two solutions $u$ and $v$ of class $C^2$ to (2.13). Then the Gaussian curvatures $K_u(x,y)$, $K_v(x,y)$ of the graphs of $u$ and $v$ at points $\big(x, y, u(x,y)\big)$ and $\big(x, y, v(x,y)\big)$, respectively, satisfy $K_u(x,y) = -K_v(x,y)$.*

**Proof.** Suppose that $q_1 = 0$. Then, by Theorem 2.10,

$$u(x,y) = \phi(x) + \psi(y)$$

and

$$v(x,y) = -\phi(x) + \psi(y),$$

for some functions $\phi$ and $\psi$ of class $C^2$. The Gaussian curvature of the graph of $u$ at $\big(x, y, u(x,y)\big)$ reads

$$K_u(x,y) = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1+u_x^2+u_y^2)^2} = \frac{\phi''(x)\psi''(y)}{\big[1+\phi'^2(x)+\psi'^2(y)\big]^2}. \tag{2.22}$$

Similarly, the Gaussian curvature of the graph of $v$ at $\big(x, y, v(x,y)\big)$ reads

$$K_v(x,y) = \frac{v_{xx}v_{yy} - v_{xy}^2}{(1+v_x^2+v_y^2)^2} = \frac{-\phi''(x)\psi''(y)}{\big[1+\phi'^2(x)+\psi'^2(y)\big]^2}.$$

It is clear that

$$K_u(x,y) = -K_v(x,y).$$

The case $q_2 = 0$ is treated analogously.

Suppose now that $q_1q_2 \neq 0$. Then, by Theorem 2.10,

$$u(x,y) = \phi(q_1x + q_2y) + \psi(q_1y - q_2x)$$

and

$$v(x,y) = \phi(q_1x + q_2y) - \psi(q_1y - q_2x),$$

for some functions $\phi$ and $\psi$ of class $C^2$. It is easily verified that the Gaussian curvature of the graph of $u$ at $\big(x, y, u(x,y)\big)$ and the Gaussian curvature of the graph of $v$ at $\big(x, y, v(x,y)\big)$ are

$$K_u(x,y) = \frac{(q_1^2+q_2^2)^2\phi''(q_1x+q_2y)\psi''(q_1y-q_2x)}{\Big\{1+(q_1^2+q_2^2)\big[\phi'^2(q_1x+q_2y)+\psi'^2(q_1y-q_2x)\big]\Big\}^2}, \tag{2.23}$$

$$K_v(x,y) = \frac{-(q_1^2+q_2^2)^2\phi''(q_1x+q_2y)\psi''(q_1y-q_2x)}{\Big\{1+(q_1^2+q_2^2)\big[\phi'^2(q_1x+q_2y)+\psi'^2(q_1y-q_2x)\big]\Big\}^2},$$

respectively. Hence

$$K_u(x,y) = -K_v(x,y). \quad \square$$

As the last corollary shows, there is no $\pm u + c$ ambiguity in two-source photometric stereo when both $\Lambda > 0$ and the Gaussian curvature of at least one solution does not vanish. However, one can expect non-uniqueness resulting from replacing $u$ by $-u + c$ for the surfaces with zero Gaussian curvature. That indeed may be the case is shown in Example 2.26. Note also that no two graphs of essentially different solutions to (2.13) are isometric unless the Gaussian curvatures of those graphs vanish at each point of $\Omega$. Since the class of surfaces with zero Gaussian curvature coincides with the class of so-called developable surfaces (see Definition 2.22), Corollary 2.11 implies also that if one of the solutions is a developable surface, then so too is the other. Example 2.4 shows that this is not the case for the wider class of so-called ruled surfaces (see Definition 2.21). The case when $\Lambda$ is merely non-negative will be treated later.

We now prove the following:

**Corollary 2.12.** *Let* $p = (0, 0, -1)$ *and* $q = (q_1, q_2, q_3)$ *be such that* $q_1^2 + q_2^2 > 0$ *and* $\|q\| = 1$. *Assume that* $\Lambda > 0$ *over an open convex* $\Omega$. *Suppose, moreover, that there exist exactly two solutions* $u$ *and* $v$ *of class* $C^2$ *to* (2.13) *with zero Gaussian curvature at each point of the graphs of* $u$ *and* $v$. *Then:*

*1. If* $q_1 q_2 = 0$ *and* $u_{yy} \neq 0$ *over* $\Omega$, *then there exist constants* $a$, $b$, *and a function* $\psi$ *of class* $C^2$ *such that* $u = u_1$, *where*

$$u_1(x, y) = ax + b + \psi(y). \tag{2.24}$$

*2. If* $q_1 q_2 = 0$ *and* $u_{xx} \neq 0$ *over* $\Omega$, *then there exist constants* $c$, $d$, *and a function* $\phi$ *of class* $C^2$ *such that* $u = u_2$, *where*

$$u_2(x, y) = \phi(x) + cy + d. \tag{2.25}$$

*3. If* $q_1 q_2 \neq 0$ *and*

$$q_1^2 u_{xx} + q_2^2 u_{yy} + 2q_1 q_2 u_{xy} \neq 0$$

*over* $\Omega$, *then there exist constants* $e$, $f$ *and a function* $\hat{\phi}$ *of class* $C^2$ *such that* $u = \hat{u}_1$, *where*

$$\hat{u}_1(x, y) = \hat{\phi}(q_1 x + q_2 y) + e(q_1 y - q_2 x) + f. \tag{2.26}$$

*4. If $q_1q_2 \neq 0$ and*

$$q_2^2 u_{xx} + q_1^2 u_{yy} - 2q_1q_2u_{xy} \neq 0$$

*over $\Omega$, then there exist some constants $l$, $k$ and a function $\hat{\psi}$ of class $C^2$ such that $u = \hat{u}_2$, where*

$$\hat{u}_2(x, y) = l(q_1x + q_2y) + k + \hat{\psi}(q_1y - q_2x). \tag{2.27}$$

*Moreover, if $u = u_1$, then*

$$v(x, y) = \begin{cases} -ax - b + \psi(y) & \text{if } q_1 = 0, \\ ax + b - \psi(y) & \text{if } q_2 = 0; \end{cases} \tag{2.28}$$

*if $u = u_2$, then*

$$v(x, y) = \begin{cases} -\phi(x) + cy + d & \text{if } q_1 = 0, \\ \phi(x) - cy - d & \text{if } q_2 = 0; \end{cases} \tag{2.29}$$

*if $u = \hat{u}_1$, then*

$$v(x, y) = e(q_1x + q_2y) + f - \hat{\psi}(q_1y - q_2x); \tag{2.30}$$

*and, if $u = \hat{u}_2$, then*

$$v(x, y) = \hat{\phi}(q_1x + q_2y) - k(q_1y - q_2x) - l. \tag{2.31}$$

**Proof.** Assume that $q_1q_2 = 0$. Then, by Theorem 2.8,

$$u(x, y) = \phi(x) + \psi(y)$$

for some functions $\phi$ and $\psi$ of class $C^2$. Since the Gaussian curvature of the graph of $u$ vanishes, we have, by (2.22), that $\phi''(x)\psi''(y) = 0$. Therefore if $u_{yy} = \psi''(y) \neq 0$, then $\phi''(x) = 0$, and so there exist constants $a$, $b$, and a function $\psi$ such that

$$u(x, y) = u_1(x, y) = ax + b + \psi(y).$$

By Theorem 2.10,

$$v(x, y) = \begin{cases} -ax - b + \psi(y) & \text{if } q_1 = 0, \\ ax + b - \psi(y) & \text{if } q_2 = 0. \end{cases}$$

Similarly, if $u_{xx} = \phi''(x) \neq 0$, then $\psi''(y) = 0$, and so there exist constants $c$, $d$, and a function $\phi$ such that

$$u(x, y) = u_2(x, y) = \phi(x) + cy + d.$$

By Theorem 2.10,

$$v(x, y) = \begin{cases} -\phi(x) + cy + d & \text{if } q_1 = 0, \\ \phi(x) - cy - d & \text{if } q_2 = 0. \end{cases}$$

Assume now that $q_1 q_2 \neq 0$. Then, by Theorem 2.10,

$$u(x, y) = \hat{\phi}(q_1 x + q_2 y) + \hat{\psi}(q_1 y - q_2 x),$$

for some functions $\hat{\phi}$ and $\hat{\psi}$ of class $C^2$. Since the Gaussian curvature of the graph of $u$ vanishes, it follows from (2.23) that

$$\hat{\phi}''(q_1 x + q_2 y)\hat{\psi}''(q_1 y - q_2 x) = 0.$$

Let $T$ be the linear transformation of $\mathbb{R}^2$ given by

$$\xi(x, y) = q_1 x + q_2 y,$$

$$\eta(x, y) = -q_2 x + q_1 y.$$

Let

$$\bar{u}(\xi, \eta) = u\big(T^{-1}(\xi, \eta)\big),$$

where $T^{-1}$ is the inverse transformation

$$x(\xi, \eta) = \frac{q_1 \xi - q_2 \eta}{q_1^2 + q_2^2},$$

$$y(\xi, \eta) = \frac{q_2 \xi + q_1 \eta}{q_1^2 + q_2^2}.$$

Clearly,

$$\bar{u}(\xi, \eta) = \phi(\xi) + \psi(\eta)$$

and

$$\bar{u}_{\xi\xi} = \phi''(\xi), \qquad \bar{u}_{\eta\eta} = \psi''(\eta).$$

Moreover,

$$\bar{u}_{\xi\xi} = \frac{1}{(q_1^2+q_2^2)^2}(q_1^2 u_{xx} + q_2^2 u_{yy} + 2q_1 q_2 u_{xy}),$$

$$\bar{u}_{\eta\eta} = \frac{1}{(q_1^2+q_2^2)^2}(q_2^2 u_{xx} + q_1^2 u_{yy} - 2q_1 q_2 u_{xy}).$$

Now proceeding along the same lines as in the case $q_1 q_2 = 0$, we obtain (2.26), (2.27), (2.30), and (2.31). □

Corollary 2.12 provides a representation theorem for a rather special class of pairs of the solutions to (2.13) with zero Gaussian curvature. There exist pairs of solutions that are not covered by this corollary. Nevertheless, even such a weak representation theorem will prove helpful in generating specific examples.

The next proposition provides, *inter alia,* a characterization of a subclass of pairs $(u, v)$ of the solutions to (2.13) with zero Gaussian curvature. Example 2.25(ii) shows that this subclass is essentially different from the class of all pairs of solutions with zero Gaussian curvature.

**Proposition 2.13.** *Let $p = (0,0,-1)$ and $q = (q_1, q_2, q_3)$ be such that $q_1^2+q_2^2 > 0$. Assume that $\Lambda > 0$ over an open convex $\Omega$. Suppose, moreover, that there exist exactly two solutions $u$ and $v$ of class $C^2$ to (2.13). Then $u = -v + c$, where $c$ is a constant, if and only if*

$$\begin{aligned} q_1 q_3 E_1(x,y) + q_1 E_2(x,y) &= 0, \\ q_2 q_3 E_1(x,y) + q_2 E_2(x,y) &= 0 \end{aligned} \tag{2.32}$$

*over $\Omega$. If $u = -v + c$, for some constant $c$, then there exist functions $\phi$ and $\psi$ of class $C^2$ such that:*

*1. If $q_1 = 0$, then*

$$u(x,y) = \phi(x) \quad \text{and} \quad v(x,y) = -\phi(x) + c.$$

*2. If $q_2 = 0$, then*

$$u(x,y) = \psi(y) \quad \text{and} \quad v(x,y) = -\psi(y) + c.$$

*3. If* $q_1 q_2 \neq 0$, *then*

$$u(x,y) = \psi(q_1 y - q_2 x) \quad \text{and} \quad v(x,y) = -\psi(q_1 y - q_2 x) + c.$$

**Proof.** The proof is an immediate consequence of Theorems 2.8 and 2.10, and Lemma 2.6 combined with the fact that $u = -v + c$. $\square$

Theorems 2.7, 2.8, and 2.10, Lemma 2.9 and Corollary 2.11 can be illustrated by the following examples.

**Example 2.14.** Let $p$, $q$, $E_1$, $E_2$, and $\Sigma$ be as specified in Example 2.4. The function $u^+(x,y) = xy$ satisfies (2.11) and also satisfies (2.14) which in this case takes the form

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{2.33}$$

Theorem 2.7 guarantees the existence of another solution $v$ of class $C^2$ to (2.11), which, by Corollary 2.2, is unique. By Theorem 2.8,

$$u^+(x,y) = \phi(x+y) + \psi(y-x),$$

where, by Lemma 2.9,

$$\phi(x) = u^+\left(\frac{1}{2}x, \frac{1}{2}x\right) - c = \frac{1}{4}x^2$$

and

$$\psi(x) = u^+\left(-\frac{1}{2}x, \frac{1}{2}x\right) + c = -\frac{1}{4}x^2.$$

By Theorem 2.10,

$$v(x,y) = \phi(x+y) - \psi(y-x) = \frac{1}{4}(x+y)^2 + \frac{1}{4}(y-x)^2 = \frac{1}{2}(x^2+y^2)$$

which obviously coincides with $u^-$ introduced in Example 2.4. Observe that $v$ also satisfies (2.33). Note, moreover, that by Corollary 2.11 the graph of $u^+$

consists exclusively of hyperbolic points (with negative Gaussian curvature), and so the graph of $u^-$ consists exclusively of elliptic points (with positive Gaussian curvature).

The next example exhibits uniqueness in two-source photometric stereo for a number of commonly encountered shapes.

**Example 2.15** Assume that $p = (0, 0, -1)$ and $q = (q_1, q_2, q_3)$ are linearly independent and that $\|q\| = 1$.

(*i*) Let

$$u(x, y) = \sqrt{R^2 - x^2 - y^2}.$$

Clearly, the graph of $u$ coincides with the upper hemisphere with radius $R$ centered at zero. The corresponding images $E_1$ and $E_2$, defined over $\Omega$ as specified at the beginning of this section, are

$$E_1(x, y) = \frac{1}{R}\sqrt{R^2 - x^2 - y^2},$$

$$E_2(x, y) = \frac{-q_1 x - q_2 y - q_3\sqrt{R^2 - x^2 - y^2}}{R}.$$

It is easy to verify that $\Lambda$, as introduced in (2.3), vanishes if and only if $q_2 x - q_1 y = 0$. Condition (2.14), which reads in this case

$$q_1 q_2 (x^2 - y^2) - (q_1^2 - q_2^2) xy = 0,$$

is not satisfied over any open subset of $\mathbb{R}^2$. By Theorem 2.7, over any open convex subset $\Sigma$ of $\Omega' = \Omega \cap \{(x, y) : q_2 x - q_1 y \neq 0\}$ there is no solution of class $C^2$ to (2.13) different from $u$. Consequently, the shape of the graph of $u$ over $\Sigma$ is uniquely determined by the images $E_1$ and $E_2$.

(*ii*) Let

$$u(x, y) = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}},$$

where $a$, $b$, and $c$ are arbitrary positive constants. Obviously, the graph of $u$ is the upper hemiellipsoid centered at zero with semiaxes $a$, $b$, and $c$. The corresponding images $E_1$ and $E_2$ over $\Omega$ are

$$E_1(x,y) = \frac{\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}}{\sqrt{1 + \dfrac{x^2}{a^2}\left(\dfrac{c^2}{a^2} - 1\right) + \dfrac{y^2}{b^2}\left(\dfrac{c^2}{b^2} - 1\right)}},$$

$$E_2(x,y) = \frac{-cq_1 b^2 x - cq_2 a^2 y - q_3 a^2 b^2 \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}}{a^2 b^2 \sqrt{1 + \dfrac{x^2}{a^2}\left(\dfrac{c^2}{a^2} - 1\right) + \dfrac{y^2}{b^2}\left(\dfrac{c^2}{b^2} - 1\right)}}.$$

It is easy to check that $\Lambda$ vanishes over $\Omega' = \{(x,y) \in \mathbb{R}^2 : q_2 b^2 x - q_1 a^2 y = 0\}$. Now condition (2.14), taking the form

$$q_1 q_2 a^2 b^2 (x^2 - y^2 + b^2 - a^2) - (q_1^2 - q_2^2) xy = 0,$$

is not satisfied over any open subset of $\mathbb{R}^2$. By Theorem 2.7, over any open convex subset $\Sigma$ of $\Omega \backslash \Omega'$ there is no solution of class $C^2$ to (2.13) different from $u$. Hence, the shape of the graph of $u$ over $\Sigma$ is uniquely determined by the images $E_1$ and $E_2$.

(*iii*) Let

$$u(x,y) = \frac{1}{2(1 + x^2 + y^2)}.$$

The graph of $u$ is mountainlike in shape (see Figure 2). The corresponding images $E_1$ and $E_2$ over $\Omega$ are

$$E_1(x,y) = \frac{(x^2 + y^2 + 1)^2}{\sqrt{x^2 + y^2 + (x^2 + y^2 + 1)^4}},$$

$$E_2(x,y) = \frac{-q_1 x - q_2 y - q_3 (x^2 + y^2 + 1)^2}{\sqrt{x^2 + y^2 + (x^2 + y^2 + 1)^4}}.$$

It is clear that $\Lambda$ vanishes over $\Omega' = \{(x,y) \in \mathbb{R}^2 : q_2 x - q_1 y = 0\}$. Condition (2.14) reads in this case

$$q_1 q_2 (y^2 - x^2) + (q_1^2 - q_2^2) xy = 0$$

and is not satisfied over any open subset of $\mathbb{R}^2$. By Theorem 2.7, over any open convex subset $\Sigma$ of $\Omega \setminus \Omega'$ there is no solution of class $C^2$ to (2.13) different from $u$. Hence, the shape of the graph of $u$ over $\Sigma$ is uniquely determined by the images $E_1$ and $E_2$.

(*iv*) Let

$$u(x, y) = \sqrt{x^2 + y^2},$$

for $x^2 + y^2 > 0$. Clearly, the graph of $u$ is a cone without peak centered at zero. The corresponding images $E_1$ and $E_2$ over $\Omega$ are

$$E_1(x, y) = \frac{1}{\sqrt{2}},$$

$$E_2(x, y) = \frac{q_1 x + q_2 y - q_3\sqrt{x^2 + y^2}}{\sqrt{2}\sqrt{x^2 + y^2}}.$$

It is easily checked that $\Lambda$ vanishes over $\Omega' = \{(x, y) \in \mathbb{R}^2 : q_2 x - q_1 y = 0\}$. Now, condition (2.14) reads

$$q_1 q_2 (x^2 - y^2) - (q_1^2 - q_2^2) x y = 0$$

and is not satisfied over any open subset of $\mathbb{R}^2$. By Theorem 2.7, over any open convex subset $\Sigma$ of $\Omega \setminus \Omega'$ there is no solution of class $C^2$ to (2.13) different from $u$. Consequently, the shape of the graph of $u$ over $\Sigma$ is uniquely determined by the images $E_1$ and $E_2$.

(*v*) Assume that $q_2 = 0$ and let

$$u(x, y) = \frac{1}{4\left[1 + (1 - x^2 - y^2)^2\right]}.$$

The graph of $u$ is volcano-like in shape (see Figure 3). The corresponding images $E_1$ and $E_2$ over $\Omega$ are

$$E_1(x,y) = \frac{\left[1+(1-x^2-y^2)^2\right]^2}{\sqrt{(x^2+y^2)(1-x^2-y^2)^2+\left[1+(1-x^2-y^2)^2\right]^4}},$$

$$E_2(x,y) = \frac{q_1 x(1-x^2-y^2) - q_3\left[1+(1-x^2-y^2)^2\right]^2}{\sqrt{(x^2+y^2)(1-x^2-y^2)^2+\left[1+(1-x^2-y^2)^2\right]^4}}.$$

It is easily checked that $\Lambda$ vanishes over $\Omega' = \{(x,y) \in \mathbb{R}^2 : y(1-x^2-y^2) = 0\}$. The condition (2.14) reads

$$xy\left[5(1-x^2-y^2)^2 - 1\right] = 0$$

and is not satisfied over any open subset of $\mathbb{R}^2$. By Theorem 2.7, over any open convex subset $\Sigma$ of $\Omega \setminus \Omega'$ there is no solution of class $C^2$ to (2.13) different from $u$. Consequently, the shape of the graph of $u$ over $\Sigma$ is uniquely determined by the images $E_1$ and $E_2$.

## 2.2. Case $\Lambda \equiv 0$

Now we shall consider the case when, for given $E_1$ and $E_2$ defined over some domain $\Omega$, $\Lambda$ introduced in Theorem 2.1 vanishes over $\Omega$. Throughout this section the assumption that $p_1 = p_2 = 0$, $p_3 = -1$, and $\|p\| = \|q\| = 1$ will be dropped.

As an immediate consequence of Theorem 2.1, we get the following:

**Corollary 2.16.** *Let $E_1$ and $E_2$ be continuous functions on a domain $\Omega$ in $\mathbb{R}^2$ with values in $[0,1]$. Suppose that $\Lambda(x,y) \equiv 0$ over $\Omega$. Then there exists at most one solution $u$ of class $C^1$ to (2.1) with partial derivatives*

$$u_x = \frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2},$$

$$u_y = \frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2}. \tag{2.34}$$

The next corollary formulates a necessary and sufficient condition for existence of exactly one solution $u$ of class $C^2$ to the system (2.1). The proof will be omitted as an elementary argument from calculus.

**Corollary 2.17.** *Let $E_1$ and $E_2$ be functions of class $C^1$ over a simply connected region $\Omega$ of $\mathbb{R}^2$ with values in $[0,1]$. Suppose that $\Lambda \equiv 0$ on $\Omega$ and that*

$$\sigma = \|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2$$

*does not vanish over $\Omega$. Then a necessary and sufficient condition for the existence of exactly one solution $u$ of class $C^2$ to (2.1) is*

$$\frac{\partial}{\partial y}\left(\frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2}\right). \tag{2.35}$$

*If this condition is fulfilled, then $u$ satisfies* (2.34).

Note that if $E_1$ and $E_2$ are derived from a genuine solution, then $\sigma$ does not vanish.

As the last corollaries show, in the case when $\Lambda$ vanishes over the entirety of $\Omega$, the shape recovery problem in two-source photometric stereo is uniquely determined. Moreover, the process of recovering such a shape is fully specified. Namely, as previously indicated, Theorem 2.1 and the formula

$$u(x,y) = \int_\gamma u_x dx + u_y dy + u(x_0, y_0),$$

where $\gamma \subset \Omega$ is a smooth curve joining points $(x, y)$ and $(x_0, y_0)$ in $\Omega$, determine a solution $u$ to (2.1). The next theorem, however, gives an alternative way of determining the function $u$ in the case when $\Lambda \equiv 0$ over $\Omega$. We start by establishing an auxiliary result.

**Lemma 2.18.** *Any solution $u$ of class $C^1$ to*

$$v_1 u_x + v_2 u_y = 0 \tag{2.36}$$

*with $v_1^2 + v_2^2 > 0$ over an open convex $\Omega$ is given by*

$$u(x, y) = \phi(v_2 x - v_1 y), \tag{2.37}$$

*where $\phi$ is an arbitrary real function of class $C^1$.*

**Proof.** Suppose $u$ is given by (2.37) for some function $\phi$ of class $C^1$. Then a direct verification shows that $u$ is a solution to (2.36).

To prove the converse, suppose that $u$ is a solution to (2.36). Fix arbitrarily $(x_0, y_0)$. Define $\phi(t)$ by

$$\phi(t) = u\left(x_0 + v_2\left(\frac{t}{v_1^2 + v_2^2} + \frac{v_1 y_0 - v_2 x_0}{v_1^2 + v_2^2}\right), y_0 - v_1\left(\frac{t}{v_1^2 + v_2^2} + \frac{v_1 y_0 - v_2 x_0}{v_1^2 + v_2^2}\right)\right).$$

We shall prove that

$$u(x, y) = \phi(v_2 x - v_1 y).$$

First observe that

$$\phi(v_2 x - v_1 y) = u\left(x_0 + v_2 \frac{v_2 x - v_1 y + v_1 y_0 - v_2 x_0}{v_1^2 + v_2^2}, y_0 - v_1 \frac{v_2 x - v_1 y + v_1 y_0 - v_2 x_0}{v_1^2 + v_2^2}\right)$$

$$= u\left(x_0 + v_2 \frac{v_2(x - x_0) - v_1(y - y_0)}{v_1^2 + v_2^2}, y_0 - v_1 \frac{v_2(x - x_0) - v_1(y - y_0)}{v_1^2 + v_2^2}\right). \tag{2.38}$$

Next note that

$$
\begin{aligned}
x_0 + v_2\frac{v_2(x-x_0) - v_1(y-y_0)}{v_1^2+v_2^2} &= \frac{x_0v_1^2 + x_0v_2^2 + v_2^2x - x_0v_2^2 - v_1v_2y + v_1v_2y_0}{v_1^2+v_2^2} \\
&= \frac{x_0v_1^2 + v_2^2x + v_1^2x - v_1^2x - v_1v_2y + v_1v_2y_0}{v_1^2+v_2^2} \\
&= \frac{x(v_1^2+v_2^2)}{v_1^2+v_2^2} + v_1\frac{v_1(x_0-x) - v_2(y-y_0)}{v_1^2+v_2^2} \\
&= x + v_1\frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2}. \qquad (2.39)
\end{aligned}
$$

We also have

$$
\begin{aligned}
y_0 - v_1\frac{v_2(x-x_0) - v_1(y-y_0)}{v_1^2+v_2^2} &= \frac{y_0v_1^2 + y_0v_2^2 - v_1v_2x + v_1v_2x_0 + v_1^2y - y_0v_1^2}{v_1^2+v_2^2} \\
&= \frac{y_0v_2^2 - v_1v_2x + v_1v_2x_0 + v_1^2y + v_2^2y - v_2^2y}{v_1^2+v_2^2} \\
&= \frac{y(v_1^2+v_2^2)}{v_1^2+v_2^2} + v_2\frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2} \\
&= y + v_2\frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2}. \qquad (2.40)
\end{aligned}
$$

Let

$$\psi(s) = u(x+v_1s, y+v_2s).$$

By (2.36)

$$\psi'(s) = v_1u_x + v_2u_y = 0,$$

which shows that $\psi$ is constant. Hence, for each $s \in \mathbb{R}$,

$$u(x,y) = \psi(0) = \psi(s).$$

In particular, if we take

$$s = \frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2},$$

then

$$
\begin{aligned}
u(x,y) &= \psi\left(\frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2}\right) \\
&= u\left(x + v_1\frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2}, y + v_2\frac{v_1(x_0-x) + v_2(y_0-y)}{v_1^2+v_2^2}\right).
\end{aligned}
$$

Taking into account (2.38), (2.39), and (2.40), we finally get

$$
\begin{aligned}
u(x,y) &= u\left(x_0 + v_2\frac{v_2(x-x_0) - v_1(y-y_0)}{v_1^2+v_2^2}, y_0 - v_1\frac{v_2(x_0-x) - v_1(y-y_0)}{v_1^2+v_2^2}\right) \\
&= \phi(v_2 x - v_1 y). \quad \square
\end{aligned}
$$

We now prove the following:

**Theorem 2.19.** *Let $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ be linearly independent vectors such that if*

$$v_1 = p_2q_3 - p_3q_2, \quad v_2 = p_3q_1 - p_1q_3, \quad \text{and} \quad v_3 = p_1q_2 - p_2q_1,$$

*then $v_1^2 + v_2^2 > 0$. Suppose, moreover, that $\Lambda \equiv 0$ over an open convex $\Omega$. If $u$ is a solution of class $C^1$ to (2.1) over $\Omega$, then*

$$
\begin{aligned}
u(x,y) &= \frac{v_3}{v_1^2+v_2^2}(v_1 x + v_2 y) \qquad (2.41)\\
&+ \int_{t_0}^{v_2x - v_1y} \frac{v_2\left[\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)\widetilde{E}_1(t) + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)\widetilde{E}_2(t)\right]}{(v_1^2+v_2^2)\left[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(t) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(t)\right]}\,dt \\
&+ \int_{t_0}^{v_2x - v_1y} \frac{v_1\left[\|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)\widetilde{E}_1(t) + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)\widetilde{E}_2(t)\right]}{(v_1^2+v_2^2)\left[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(t) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(t)\right]}\,dt,
\end{aligned}
$$

*where*

$$\widetilde{E}_1(t) = E_1\left(\frac{v_2 t}{v_1^2+v_2^2}, \frac{-v_1 t}{v_1^2+v_2^2}\right) \quad \text{and} \quad \widetilde{E}_2(t) = E_2\left(\frac{v_2 t}{v_1^2+v_2^2}, \frac{-v_1 t}{v_1^2+v_2^2}\right).$$

**Proof.** We retain the notation from Theorem 2.1. By (2.3), (2.4), (2.7), and (2.8), the unit normal $\nu$ to the graph of $u$ can be represented as

$$\nu = \alpha\widetilde{e} + \beta\widetilde{f} + \gamma\widetilde{g},$$

where

$$\gamma = \varepsilon\sqrt{\frac{\Lambda}{\|p\|^2\|q\|^2 - \langle p|q\rangle^2}}.$$

Since $\Lambda \equiv 0$ over $\Omega$, it follows that $\gamma \equiv 0$, and so $\langle \nu | \widetilde{g} \rangle = 0$ over $\Omega$. Taking into account that

$$\nu = \left( \frac{u_x}{\sqrt{u_x^2 + u_y^2 + 1}}, \frac{u_y}{\sqrt{u_x^2 + u_y^2 + 1}}, \frac{-1}{\sqrt{u_x^2 + u_y^2 + 1}} \right)$$

and

$$\widetilde{g} = \frac{1}{\|p\|^2 \|q\|^2 - \langle p|q \rangle^2} (v_1, v_2, v_3),$$

we see that

$$v_1 \frac{\partial u}{\partial x} + v_2 \frac{\partial u}{\partial y} = v_3. \tag{2.42}$$

Clearly

$$v(x, y) = u(x, y) - \frac{v_3}{v_1^2 + v_2^2} (v_1 x + v_2 y)$$

satisfies

$$v_1 \frac{\partial v}{\partial x} + v_2 \frac{\partial v}{\partial y} = 0.$$

By Lemma 2.18, there is a function $\phi$ of class $C^2$ such that

$$v(x, y) = \phi(v_2 x - v_1 y),$$

and so

$$u(x, y) = \phi(v_2 x - v_1 y) + \frac{v_3}{v_1^2 + v_2^2} (v_1 x + v_2 y). \tag{2.43}$$

To express $\phi$ in terms of $E_1$, $E_2$, $p$, and $q$ we invoke Corollary 2.16 assuring that $u_x$ and $u_y$ are given by (2.34). By (2.43),

$$u_x = v_2 \phi'(v_2 x - v_1 y) + \frac{v_1 v_3}{v_1^2 + v_2^2},$$
$$u_y = -v_1 \phi'(v_2 x - v_1 y) + \frac{v_2 v_3}{v_1^2 + v_2^2}.$$

This jointly with (2.34) yields

$$v_2 \phi'(v_2 x - v_1 y) = \frac{-v_1 v_3}{v_1^2 + v_2^2} + \frac{\|p\|(q_1 \langle p|q \rangle - p_1 \|q\|^2) E_1 + \|q\|(p_1 \langle p|q \rangle - q_1 \|p\|^2) E_2}{\|p\|(p_3 \|q\|^2 - q_3 \langle p|q \rangle) E_1 + \|q\|(q_3 \|p\|^2 - p_3 \langle p|q \rangle) E_2},$$

$$v_1 \phi'(v_2 x - v_1 y) = \frac{v_2 v_3}{v_1^2 + v_2^2} - \frac{\|p\|(q_2 \langle p|q \rangle - p_2 \|q\|^2) E_1 + \|q\|(p_2 \langle p|q \rangle - q_2 \|p\|^2) E_2}{\|p\|(p_3 \|q\|^2 - q_3 \langle p|q \rangle) E_1 + \|q\|(q_3 \|p\|^2 - p_3 \langle p|q \rangle) E_2}.$$

Before proceeding further, we present the following lemma whose proof will be given in Appendix 3.

**Lemma 2.20.** *Let*

$$v_1 = p_2q_3 - p_3q_2, \quad v_2 = p_3q_1 - p_1q_3, \quad \text{and} \quad v_3 = p_1q_2 - p_2q_1$$

*be such that* $v_1^2 + v_2^2 > 0$. *Suppose that* $E_1$ *and* $E_2$ *are continuous functions over an open convex* $\Omega$ *and that*

$$v_2\phi'(v_2x - v_1y) = \frac{-v_1v_3}{v_1^2+v_2^2} + \frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2},$$

$$v_1\phi'(v_2x - v_1y) = \frac{v_2v_3}{v_1^2+v_2^2} - \frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2}.$$

*Let*

$$\widetilde{E}_1(s) = E_1\Big(\frac{v_2s}{v_1^2+v_2^2}, \frac{-v_1s}{v_1^2+v_2^2}\Big) \quad \text{and} \quad \widetilde{E}_2(s) = E_2\Big(\frac{v_2s}{v_1^2+v_2^2}, \frac{-v_1s}{v_1^2+v_2^2}\Big).$$

*Then* $\phi$, *up to a constant, is given by*

$$\phi(t) = \int_{t_0}^{t} \frac{v_2\big[\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)\widetilde{E}_1(s) + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)\widetilde{E}_2(s)\big]}{(v_1^2+v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(s)\big]}\,ds \tag{2.44}$$
$$+ \int_{t_0}^{t} \frac{v_1\big[\|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)\widetilde{E}_2(s)\big]}{(v_1^2+v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(s)\big]}\,ds.$$

By Lemma 2.20 and (2.43), the function $u$ is given, up to a constant, by

$$u(x,y) = \frac{v_3}{v_1^2+v_2^2}(v_1x + v_2y)$$

$$+ \int_{t_0}^{v_2x - v_1y} \frac{v_2\big[\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)\widetilde{E}_1(t) + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)\widetilde{E}_2(t)\big]}{(v_1^2+v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(t) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(t)\big]}\,dt$$

$$+ \int_{t_0}^{v_2x - v_1y} \frac{v_1\big[\|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)\widetilde{E}_1(t) + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)\widetilde{E}_2(t)\big]}{(v_1^2+v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(t) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(t)\big]}\,dt.$$

□

The subsequent discussion will be devoted to a geometrical characterization of the unique solution to (2.1) in the case when $\Lambda$ vanishes. Note that, as an inspection of the proof of Theorem 2.1 reveals, $\Lambda = 0$ if and only if the normal vector field to the graph of $u$ is in the $pq$-plane. We start by recalling the following definitions (see [9, pp. 188-194]).

**Definition 2.21.** *A smooth surface in $\mathbb{R}^3$ is called a ruled surface if it can be represented in the form*

$$X(s,t) = \alpha(t) + sw(t),$$

*where $t \to \alpha(t)$ and $t \to w(t)$ are smooth mappings from $\mathbb{R}$ onto $\mathbb{R}^3$. Each line of the form*

$$L_t = \{s \in \mathbb{R} : X(s,t) = \alpha(t) + sw(t)\}$$

*is called a ruling and the curve $t \to \alpha(t)$ is called the directrix.*

Obviously the parametric representation above may not be unique. Thus the directrix and rulings are not uniquely determined.

**Definition 2.22.** *A ruled surface is said to be developable if*

$$\langle w(t) \times w'(t) | \alpha(t) \rangle = 0.$$

*for each $t \in \mathbb{R}$.*

There are two kinds of developable surfaces. A ruled surface is said to be of *cylindrical type* if the function $t \to w(t)$ in the above parametrization is constant (see Figure 4). A ruled surface is said to be of *conical type* if the function $t \to w(t)$ in the above parametrization is not constant (see Figure 5).

The next theorem will give a geometrical characterization of the graph of a unique solution to (2.1) satisfying the assumption of Theorem 2.19.

**Theorem 2.23.** *Let $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ be linearly independent vectors such that if*

$$v_1 = p_2q_3 - p_3q_2, \quad v_2 = p_3q_1 - p_1q_3, \quad \text{and} \quad v_3 = p_1q_2 - p_2q_1,$$

*then $v_1^2 + v_2^2 > 0$. Suppose, moreover, that $\Lambda \equiv 0$ over an open convex $\Omega$. Let $E_1$ and $E_2$ be continuous functions on $\Omega$, $u$ be a solution to (2.1), and $\phi$ be given by (2.44). Then the graph $\mathcal{G}(u)$ of $u$ is a developable surface of cylindrical type with parametrization*

$$(s,t) \to \alpha(t) + sw(t),$$

*where*

$$\alpha(t) = \Big(\frac{v_2 t}{v_1^2 + v_2^2}, \frac{-v_1 t}{v_1^2 + v_2^2}, \phi(t)\Big)$$

*and*

$$w(t) = \Big(\frac{v_1}{v_1^2 + v_2^2}, \frac{v_2}{v_1^2 + v_2^2}, \frac{v_3}{v_1^2 + v_2^2}\Big).$$

**Proof.** Let

$$\begin{aligned} s(x,y) &= v_1 x + v_2 y, \\ t(x,y) &= v_2 x - v_1 y. \end{aligned} \tag{2.45}$$

The inverse transformation is

$$\begin{aligned} x(s,t) &= \frac{v_2 t + v_1 s}{v_1^2 + v_2^2}, \\ y(s,t) &= \frac{v_2 s - v_1 t}{v_1^2 + v_2^2}. \end{aligned} \tag{2.46}$$

By (2.43) and (2.45),

$$u\big(x(s,t), y(s,t)\big) = \phi(t) + \frac{v_3}{v_1^2 + v_2^2} s.$$

Hence, by (2.46),

$$\big(x, y, u(x,y)\big) = \Big(\frac{v_2 t + v_1 s}{v_1^2 + v_2^2}, \frac{v_2 s - v_1 t}{v_1^2 + v_2^2}, \phi(t) + \frac{v_3}{v_1^2 + v_2^2} s\Big).$$

We thus see that $\mathcal{G}(u)$ can be parametrized in the form $(s,t) \to \alpha(t) + s w(t)$, where

$$\alpha(t) = \begin{pmatrix} \dfrac{v_2 t}{v_1^2 + v_2^2} \\ \\ \dfrac{-v_1 t}{v_1^2 + v_2^2} \\ \\ \phi(t) \end{pmatrix} \quad \text{and} \quad w(t) = \begin{pmatrix} \dfrac{v_1}{v_1^2 + v_2^2} \\ \\ \dfrac{v_2}{v_1^2 + v_2^2} \\ \\ \dfrac{v_3}{v_1^2 + v_2^2} \end{pmatrix}.$$

Correspondingly, $\mathcal{G}(u)$ is a ruled surface. Observe that since $t \to w(t)$ is constant, we have

$$\langle w(t) \times w'(t) | \alpha(t) \rangle \equiv 0,$$

which shows that $\mathcal{G}(u)$ is a developable surface of cylindrical type. $\square$

Theorems 2.19 and 2.23 and Corollaries 2.16 and 2.17 can be illustrated by the following examples.

**Example 2.24.** Let $p = (0,0,-1)$ and $q = (1,0,-1)$, and let $E_1(x,y) = (1 - x^2)^{1/2}$ and $E_2(x,y) = \big[(1-x^2)^{1/2} - x\big]/\sqrt{2}$. Consider the corresponding image irradiance equations

$$\begin{aligned} \frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} &= \sqrt{1 - x^2}, \\ \frac{u_x + 1}{\sqrt{2}\sqrt{u_x^2 + u_y^2 + 1}} &= \frac{\sqrt{1-x^2} - x}{\sqrt{2}} \end{aligned} \tag{2.47}$$

over an open convex subset $\Omega$ of $\Sigma = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1/\sqrt{2}\}$. It is easy to verify that $\Lambda \equiv 0$ over $\Omega$. Clearly, $\sigma = -(1-x^2)^{1/2}$ does not vanish over $\Omega$.

By Corollary 2.16, there exists at most one solution $u$ to (2.47) which, by (2.34), satisfies

$$u_x = \frac{-x}{\sqrt{1-x^2}} \quad \text{and} \quad u_y = 0.$$

It now immediately follows that $u(x,y) = (1-x^2)^{1/2} + c$, where $c$ is an arbitrary constant. In view of (2.35) and Corollary 2.17, $u$ is of class $C^2$.

An alternative way of finding $u$ is to apply Theorem 2.19. Note that $v_1 = 0$, $v_2 = -1$, $v_3 = 0$, and so (2.41) yields

$$u(x,y) = -\int_{t_0}^{-x} \frac{t}{\sqrt{1-t^2}}\, dt = \sqrt{1-x^2} + c.$$

To describe geometrically the graph of $u$, note that, by Lemma 2.20,

$$\phi(t) = \sqrt{1-t^2},$$

up to a constant. Thus, by Theorem 2.23, the graph of $u$ can be represented as a developable surface of cylindrical type with parametrization

$$(s,t) \to \alpha(t) + s w(t),$$

where

$$\alpha(t) = \left(-t, 0, \sqrt{1-t^2}\right)$$

and

$$w(t) = (0,-1,0)$$

(see Figure 6).

**Example 2.25.** Let $p = (0,0,-1)$ and $q = (q_1, q_2, -1)$ with $q_1^2 + q_2^2 > 0$. Then the plane formed by the graph of the function

$$u(x,y) = ax + by + c$$

generates two images $E_1$ and $E_2$ given over $\mathbb{R}^2$ by

$$E_1(x,y) = \frac{1}{\sqrt{a^2+b^2+1}},$$

$$E_2(x,y) = \frac{aq_1 + bq_2 + 1}{\sqrt{q_1^2+q_2^2+1}\sqrt{a^2+b^2+1}}.$$

Assume that $aq_1+bq_2+1>0$ and consider the following system of image irradiance equations over $\mathbb{R}^2$:

$$\frac{1}{\sqrt{u_x^2+u_y^2+1}} = \frac{1}{\sqrt{a^2+b^2+1}},$$
$$\frac{q_1u_x+q_2u_y+1}{\sqrt{q_1^2+q_2^2+1}\sqrt{u_x^2+u_y^2+1}} = \frac{aq_1+bq_2+1}{\sqrt{q_1^2+q_2^2+1}\sqrt{a^2+b^2+1}}. \tag{2.48}$$

(i) Suppose first that

$$\begin{vmatrix} 0 & q_1 & a \\ 0 & q_2 & b \\ -1 & -1 & -1 \end{vmatrix} = aq_2 - bq_1 = 0.$$

Clearly, $\Lambda \equiv 0$ over $\Omega$ and $\sigma = -(q_1^2+q_2^2)(a^2+b^2+1)^{-1/2} \neq 0$. By Corollary 2.16, $u(x,y) = ax+by+c$ is the only solution to (2.48). Consequently, the shape of a plane whose normal lies in the plane spanned by the light-source vectors $p$ and $q$ is uniquely determined by its images. Of course, such a plane is a developable surface; this fact can be independently inferred from Theorem 2.23.

(ii) Suppose now that

$$\begin{vmatrix} 0 & q_1 & a \\ 0 & q_2 & b \\ -1 & -1 & -1 \end{vmatrix} = aq_2 - bq_1 \neq 0. \tag{2.49}$$

Clearly, by (2.12), (2.48), and (2.49),

$$\Lambda = \frac{(aq_2-bq_1)^2}{a^2+b^2+1} > 0$$

and $\sigma^{\pm} = -(q_1^2+q_2^2)(a^2+b^2+1)^{-1/2} \neq 0$. Without loss of generality one may assume that $aq_2 - bq_1 > 0$. By Corollary 2.2, the equations (2.9) take the form

$$u_x^+ = a, \qquad u_y^+ = b,$$

and

$$u_x^- = \frac{a(q_1^2-q_2^2)+2bq_1q_2}{q_1^2+q_2^2} = A, \quad u_y^- = \frac{b(q_2^2-q_1^2)+2aq_1q_2}{q_1^2+q_2^2} = B.$$

The integrability condition (2.10) is obviously satisfied for each choice of sign appearing therein. By Corollary 2.3, there exist exactly two solutions $u^+$ and $u^-$ of class $C^2$ to (2.48). A straightforward calculation shows that

$$u^+(x,y) = ax + by + c \quad \text{and} \quad u^-(x,y) = Ax + By + C.$$

We thus see that the shape of a plane whose normal does not lie in the plane spanned by the vectors $p$ and $q$ is not uniquely determined. Note also that the graphs of $u^+$ and $u^-$ have zero Gaussian curvature, whereas in general $u^+ \neq u^- +$ const as, in general, condition (2.32) is not satisfied.

## 2.3. Case $\Lambda \geq 0$

So far we have considered the cases in which $\Lambda$ is either positive or vanishes over a given domain $\Omega$. Now we shall treat the situation in which $\Lambda$ is non-negative. Our analysis will not be complete, as we shall confine ourselves to the simplest cases concerning the topology of the zero set of $\Lambda$.

Assume that there exists at least one solution $u$ of class $C^2$ to (2.1) and that the functions $E_1$, $E_2$ appearing in these equations are continuous over $\Omega$. Suppose that the set

$$\{(x,y) \in \Omega : \Lambda = 0\}$$

is a smooth curve $\Gamma$ such that $\Omega \setminus \Gamma = D_1 \cup D_2$, where $D_1$ and $D_2$ are disjoint open subsets of $\Omega$ (on which, of course, $\Lambda$ is positive). By Corollary 2.2, there exist at most two solutions $(u_1^1, u_1^2)$ to (2.1) of class $C^2$ over $D_1$ and at most two solutions $(u_2^1, u_2^2)$ to (2.1) of class $C^2$ over $D_2$, respectively. We do not exclude the possibility that $u_i^1 = u_i^2$ for either $i = 1$; or $i = 2$; or $i = 1$ and $i = 2$. Clearly, the restriction of $u$ to $D_i$ coincides with either $u_i^1$ or $u_i^2$ for $i = 1, 2$. Conversely, suppose that for some $i$ and $j$ $(i, j = 1, 2)$ and some constant $c$

$$\lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_1^i(x',y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} u_2^j(x',y') + c = g^{ij}(x,y)$$

and

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{\partial u_1^i(x',y')}{\partial x} = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{\partial u_2^j(x',y')}{\partial x} = \delta^{ij}(x,y),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{\partial u_1^i(x',y')}{\partial y} = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{\partial u_2^j(x',y')}{\partial y} = \theta^{ij}(x,y),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{\partial^2 u_1^i(x',y')}{\partial x^2} = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{\partial^2 u_2^j(x',y')}{\partial x^2} = \rho^{ij}(x,y),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{\partial^2 u_1^i(x',y')}{\partial y^2} = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{\partial^2 u_2^j(x',y')}{\partial y^2} = \omega^{ij}(x,y),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{\partial^2 u_1^i(x',y')}{\partial x\partial y} = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{\partial^2 u_2^j(x',y')}{\partial x\partial y} = \tau^{ij}(x,y)$$

exist for each $(x,y)\in\Gamma$. Set

$$v_{ij}(x,y) = \begin{cases} u_1^i(x,y) & \text{if } (x,y)\in D_1, \\ g^{ij}(x,y) & \text{if } (x,y)\in\Gamma, \\ u_2^j(x,y)+c & \text{if } (x,y)\in D_2, \end{cases} \tag{2.50}$$

and suppose that for each $(x,y)\in\Gamma$

$$\frac{\partial v_{ij}(x,y)}{\partial x} = \delta^{ij}(x,y),$$

$$\frac{\partial v_{ij}(x,y)}{\partial y} = \theta^{ij}(x,y),$$

$$\frac{\partial^2 v_{ij}(x,y)}{\partial x^2} = \rho^{ij}(x,y),$$

$$\frac{\partial^2 v_{ij}(x,y)}{\partial y^2} = \omega^{ij}(x,y),$$

$$\frac{\partial^2 v_{ij}(x,y)}{\partial x \partial y} = \frac{\partial^2 v_{ij}(x,y)}{\partial y \partial x} = \tau^{ij}(x,y).$$

Then $v_{ij}$ is a solution of class $C^2$ to (2.1) over $\Omega$ and in such a case we say that the functions $u_1^i$ and $u_2^j$ *bifurcate* along $\Gamma$ in the class $C^2$. It is clear that, up to a constant, one can define in this way at most four solutions of class $C^2$ to (2.1) over $\Omega$. Obviously, the case of three solutions cannot occur (see Corollary 2.31), but, as we shall show below, in specific situations there may exist one, two, or four solutions.

**Example 2.26.** Let $p = (0,0,-1)$, $q = (0, 1/\sqrt{2}, -1/\sqrt{2})$, $E_1(x,y) = (1+x^6)^{-1/2}$, and $E_2(x,y) = \left[2(1+x^6)\right]^{-1/2}$. Consider the corresponding image irradiance equations

$$\begin{aligned} \frac{1}{\sqrt{u_x^2+u_y^2+1}} &= \frac{1}{\sqrt{1+x^6}}, \\ \frac{1+u_y}{\sqrt{2}\sqrt{u_x^2+u_y^2+1}} &= \frac{1}{\sqrt{2}\sqrt{1+x^6}} \end{aligned} \tag{2.51}$$

over any simply connected $\Omega \subset \mathbb{R}^2$ containing the line $x = 0$. Clearly

$$\Lambda = \frac{1}{2}\left(1 - \frac{1}{1+x^6}\right) \geq 0.$$

Let $\Omega = D_1 \cup D_2 \cup \Gamma$, where

$$D_1 = \{(x,y) \in \Omega : x > 0\},$$

$$D_2 = \{(x,y) \in \Omega : x < 0\},$$

$$\Gamma = \{(x,y) \in \Omega : x = 0\}.$$

By Corollaries 2.2 and 2.3,

$$u_1^1(x,y) = \frac{x^4}{4} \quad \text{and} \quad u_1^2(x,y) = -\frac{x^4}{4}$$

over $D_1$, and

$$u_2^1(x,y) = \frac{x^4}{4} \quad \text{and} \quad u_2^2(x,y) = -\frac{x^4}{4}$$

over $D_2$. Let

$$v_{11}(x,y) = \frac{x^4}{4}, \qquad v_{22}(x,y) = -\frac{x^4}{4},$$

$$v_{12}(x,y) = \begin{cases} \frac{x^4}{4} & \text{if } (x,y) \in D_1, \\ 0 & \text{if } (x,y) \in \Gamma, \\ -\frac{x^4}{4} & \text{if } (x,y) \in D_2, \end{cases}$$

and

$$v_{21}(x,y) = \begin{cases} -\frac{x^4}{4} & \text{if } (x,y) \in D_1;, \\ 0 & \text{if } (x,y) \in \Gamma, \\ \frac{x^4}{4} & \text{if } (x,y) \in D_2. \end{cases}$$

It is a matter of simple calculation to verify that the functions $v_{11}$, $v_{12}$, $v_{21}$ and $v_{22}$ are of class $C^2$ over $\Omega$. Hence there are exactly four solutions of class $C^2$ to (2.51) over $\Omega$. Note that each solution $u$ generates another solution of the form $-u+c$, where $c$ is a constant. As was shown in Corollary 2.11, this is only possible when the Gaussian curvature vanishes, which here is the case (the graphs of all solutions are developable surfaces). Note also that $E_1$ and $E_2$ satisfy (2.32).

**Example 2.27.** Let vectors $p$, $q$ and functions $E_1$, $E_2$ be defined as in Example 2.4, and let $\Omega$ be a simply connected domain of $\mathbb{R}^2$ containing the line $x = y$. Let

$$D_1 = \{(x,y) : x < y\},$$
$$D_2 = \{(x,y) : x > y\},$$
$$\Gamma = \{(x,y) : x = y\}.$$

As was shown in Example 2.4, $\Lambda$ vanishes along $x = y$ and moreover

$$u_1^1(x,y) = xy \quad \text{and} \quad u_1^2(x,y) = \frac{x^2+y^2}{2}$$

over $D_1$, and

$$u_2^1(x,y) = xy \quad \text{and} \quad u_2^2(x,y) = \frac{x^2+y^2}{2}$$

over $D_2$. Let

$$v_{11}(x,y) = xy, \qquad v_{22}(x,y) = \frac{x^2+y^2}{2},$$

$$v_{12}(x,y) = \begin{cases} xy & \text{if } (x,y) \in D_1, \\ x^2 & \text{if } (x,y) \in \Gamma, \\ \frac{x^2+y^2}{2} & \text{if } (x,y) \in D_2, \end{cases}$$

and

$$v_{21}(x,y) = \begin{cases} \frac{x^2+y^2}{2} & \text{if } (x,y) \in D_1, \\ x^2 & \text{if } (x,y) \in \Gamma, \\ xy & \text{if } (x,y) \in D_2. \end{cases}$$

It is easy to verify that only the first two functions are of class $C^2$ over $\Omega$. Hence there are exactly two solutions of class $C^2$ to (2.11) over $\Omega$.

**Example 2.28.** In all cases considered in Example 2.15, $\Lambda$ vanishes along some line in $\mathbb{R}^2$. Consider a convex open $\Omega$ containing the line where $\Lambda$ vanishes. Then

$$\Omega = D_1 \cup D_2 \cup \Gamma,$$

where $D_1$, $D_2$, and $\Gamma$ are defined as in Example 2.26 and 2.27. As was shown in Example 2.15, there is only one solution $u = u_1^1 = u_1^2$ of class $C^2$ over $D_1$, and only one solution $u = u_2^1 = u_2^2$ of class $C^2$ over $D_2$. Hence there exists at most one solution over $\Omega$ given by (2.50). An easy verification shows that (2.50) defines a solution of class $C^2$ to (2.1). We thus see that hemispheres, hemiellipsoids, cones, volcano-like surfaces and mountainlike surfaces are uniquely determined by two-source photometric stereo.

Now we shall show that, under the assumption that $\Lambda \geq 0$, if the set $\Omega \backslash \Gamma$ has more than two components, then the number of solutions to (2.1) may be greater than four.

**Example 2.29.** Let $p = (0,0,-1)$ and $q = (0,1,-1)$. Consider the functions

$$u_0(x,y) = \frac{x^3}{3}\sin^3\frac{1}{x} \qquad \text{and} \qquad u_1(x,y) = -\frac{x^3}{3}\sin^3\frac{1}{x}$$

over $\Omega = (0, 1/\pi)\times\mathbb{R}$. Clearly, for all $(x,y)\in\Omega\backslash\bigcup_{n=2}^{\infty}\Gamma_n$, where $\Gamma_n = \{1/n\pi\}\times\mathbb{R}$ both functions $u_0$ and $u_1$ are of class $C^2$. It is also easy to verify that $u_0$ and $u_1$ are functions of class $C^2$ in an open neighbourhood of $\bigcup_{n=2}^{\infty}\Gamma_n$ and moreover $u_{ix} = u_{iy} = 0$ on $\bigcup_{n=2}^{\infty}\Gamma_n$ for $i = 0, 1$. The corresponding images $E_1$ and $E_2$ defined over $\Omega$ are

$$E_1(x,y) = \left[1 + \left(x^2\sin^3\frac{1}{x} - x\cos\frac{1}{x}\sin^2\frac{1}{x}\right)^2\right]^{-1/2},$$

$$E_2(x,y) = \left\{2\left[1 + \left(x^2\sin^3\frac{1}{x} - x\cos\frac{1}{x}\sin^2\frac{1}{x}\right)^2\right]\right\}^{-1/2}.$$

Consider now the following system of two image irradiance equations:

$$\begin{aligned} \frac{1}{\sqrt{u_x^2 + u_y^2 + 1}} &= E_1(x,y), \\ \frac{u_y + 1}{\sqrt{2}\sqrt{u_x^2 + u_y^2 + 1}} &= E_2(x,y). \end{aligned} \tag{2.52}$$

It is easy to verify that $\Lambda(1/n\pi, y) = 0$ for each $n \geq 2$. Let $\Omega_2 = \big\{(x,y)\in\Omega : 1/3\pi < x < 1/\pi\big\}$, and let

$$v_{00}(x,y) = \begin{cases} \frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{2\pi} < x < \frac{1}{\pi}, \\ 0 & \text{if } x = \frac{1}{2\pi}, \\ \frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{3\pi} < x < \frac{1}{2\pi}, \end{cases}$$

$$v_{01}(x,y) = \begin{cases} \frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{2\pi} < x < \frac{1}{\pi}, \\ 0 & \text{if } x = \frac{1}{2\pi}, \\ -\frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{3\pi} < x < \frac{1}{2\pi}, \end{cases}$$

$$v_{10}(x,y) = \begin{cases} -\frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{2\pi} < x < \frac{1}{\pi}, \\ 0 & \text{if } x = \frac{1}{2\pi}, \\ \frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{3\pi} < x < \frac{1}{2\pi}, \end{cases}$$

$$v_{11}(x,y) = \begin{cases} -\frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{2\pi} < x < \frac{1}{\pi}, \\ 0 & \text{if } x = \frac{1}{2\pi}, \\ -\frac{x^3}{3}\sin^3\frac{1}{x} & \text{if } \frac{1}{3\pi} < x < \frac{1}{2\pi}. \end{cases}$$

It is easy to verify that functions $v_{00}$, $v_{01}$, $v_{10}$, and $v_{11}$ are solutions of class $C^2$ to (2.52) over $\Omega_2$. Proceeding by induction on $n$, given an integer $n > 2$ and $(i_1, i_2, ..., i_{n-1}) \in \{0,1\}^{n-1}$, let $v_{i_1 i_2 \ldots i_{n-1}}$ be the solution to (2.52) over $\Omega_{n-1} = (1/n\pi, 1/\pi) \times \mathbb{R}$ constructed in the $n-1$ step. For each $(i_1, ..., i_{n-1}, i_n) \in \{0,1\}^n$, set

$$v_{i_1 \ldots i_{n-1} i_n}(x,y) = \begin{cases} v_{i_1 \ldots i_{n-1}}(x,y) & \text{if } \frac{1}{n\pi} < x < \frac{1}{\pi}, \\ 0 & \text{if } x = \frac{1}{n\pi}, \\ u_{i_n}(x,y) & \text{if } \frac{1}{(n+1)\pi} < x < \frac{1}{n\pi}. \end{cases}$$

Clearly, each $v_{i_1 \ldots i_n}$ is a $C^2$ class solution to (2.52) over $\Omega_n$, and so there are infinitely many bounded solutions of class $C^2$ to (2.52) over $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ possessing a continuous extension to the $y$-axis.

The last examples indicate that bifurcation in the class of $C^2$ functions may or may not take place along the curve $\Gamma$, where $\Lambda$ vanishes. In consequence, the feasibility of gluing two solutions to (2.13) along $\Gamma$ affects global existence and uniqueness results for two-image shape recovery. The next results establish necessary and sufficient conditions for such a bifurcation to take place.

For the sake of brevity, the next theorem concerns only the case in which one of the two light-source vectors is overhead and both vectors are normalized. It can, however, easily be extended to the case of an arbitrary pair of linearly independent vectors.

**Corollary 2.30.** *Let $p = (0,0,-1)$ and let $q = (q_1, q_2, q_3)$ be such that $q_1^2 + q_2^2 > 0$ and $\|q\| = 1$. Let $E_1$ and $E_2$ be continuous functions such that the set $\{(x,y) \in \Omega : \Lambda = 0\}$ is a smooth curve $\Gamma$. Suppose that $\Omega \setminus \Gamma = D_1 \cup D_2$, where $D_1$ and $D_2$ are disjoint open subsets of $\Omega$. Suppose, moreover, that for a pair of solutions*

*$(u, v)$ of class $C^1$ to (2.13) defined over $D_1$ and $D_2$, respectively,*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u(x',y') = \bar{u}(x,y), \qquad \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v(x',y') = \bar{v}(x,y), \tag{2.53}$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\big(u_x(x',y'), u_y(x',y')\big), \quad \text{and} \quad \lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\big(v_x(x',y'), v_y(x',y')\big)$$

*exist. Then there exists a constant $c$ such that the function*

$$z(x,y) = \begin{cases} u(x,y) & \text{if } (x,y)\in D_1,\\ \bar{u}(x,y) & \text{if } (x,y)\in\Gamma,\\ v(x,y)+c & \text{if } (x,y)\in D_2 \end{cases}$$

*is a solution to (2.13) of class $C^1$ over $\Omega$.*

**Proof.** First we show that for each $(x,y)\in\Gamma$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\big(u_x(x',y'), u_y(x',y')\big) = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\big(v_x(x',y'), v_y(x',y')\big). \tag{2.54}$$

Let

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\big(u_x(x',y'), u_y(x',y')\big) = (r_1, r_2), \tag{2.55}$$

$$\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\big(v_x(x',y'), v_y(x',y')\big) = (l_1, l_2). \tag{2.56}$$

Since $u$ and $v$ satisfy (2.13) over $D_1$ and $D_2$, it follows that

$$\begin{aligned} \lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{1}{\sqrt{u_x^2+u_y^2+1}} &= \lim_{(x',y')\in D_1\to(x,y)\in\Gamma} E_1(x',y'),\\ \lim_{(x',y')\in D_1\to(x,y)\in\Gamma} \frac{q_1u_x+q_2u_y-q_3}{\sqrt{u_x^2+u_y^2+1}} &= \lim_{(x',y')\in D_1\to(x,y)\in\Gamma} E_2(x',y') \end{aligned} \tag{2.57}$$

and

$$\begin{aligned} \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{1}{\sqrt{v_x^2+v_y^2+1}} &= \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} E_1(x',y'),\\ \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} \frac{q_1v_x+q_2v_y-q_3}{\sqrt{v_x^2+v_y^2+1}} &= \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} E_2(x',y'). \end{aligned} \tag{2.58}$$

Combining (2.55) and (2.56) with (2.57) and (2.58), we get

$$\frac{1}{\sqrt{r_1^2+r_2^2+1}} = E_1(x,y),$$
$$\frac{q_1r_1+q_2r_2-q_3}{\sqrt{r_1^2+r_2^2+1}} = E_2(x,y) \tag{2.59}$$

and

$$\frac{1}{\sqrt{l_1^2+l_2^2+1}} = E_1(x,y),$$
$$\frac{q_1l_1+q_2l_2-q_3}{\sqrt{l_1^2+l_2^2+1}} = E_2(x,y). \tag{2.60}$$

Now (2.59) implies that

$$q_1r_1+q_2r_2 = \widetilde{E}(x,y), \tag{2.61}$$

where

$$\widetilde{E}(x,y) = \frac{E_2(x,y)}{E_1(x,y)} + q_3,$$

and (2.60) yields

$$q_1l_1+q_2l_2 = \widetilde{E}(x,y). \tag{2.62}$$

Note that since functions $E_1$ and $E_2$ are continuous, it follows by (2.3) that $\Lambda$ is continuous. Therefore,

$$\lim_{(x',y')\in\Omega\to(x,y)\in\Gamma} \Lambda(x',y') = \Lambda(x,y) = 0.$$

This together with Lemma 2.5 yields

$$q_2r_1 = q_1r_2 \quad \text{and} \quad l_1q_2 = l_2q_1. \tag{2.63}$$

By multiplying (2.61) and (2.62) by $q_2$ and taking into account (2.63), we get

$$(q_1^2+q_2^2)r_2 = q_2\widetilde{E}(x,y),$$
$$(q_1^2+q_2^2)l_2 = q_2\widetilde{E}(x,y).$$

Hence, clearly, $r_2 = l_2$.

Similarly, multiplying (2.61) and (2.62) by $q_2$ and taking into account (2.63), we easily conclude that $r_1 = l_1$. This completes the proof of (2.54).

We now claim that the function $h = \bar{v} - \bar{u}$ is constant. Let $A$ and $B$ be two different points in $\Gamma$. Let $[0,1] \ni t \to \gamma(t) = \big(x(t), y(t)\big)$ be a $C^1$ class parametrization of $\Gamma$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Let $A_n$ and $B_n$ in $D_1$ be such that

$$\lim_{n\to\infty} A_n = A \quad \text{and} \quad \lim_{n\to\infty} B_n = B. \tag{2.64}$$

For each $n \in \mathbb{N}$, let $\gamma_n$ be a smooth curve in $D_1$ joining $A_n$ and $B_n$ with $C^1$ class parametrization $[0,1] \ni t \to \gamma_n(t) = \big(x_n(t), y_n(t)\big)$ such that $\gamma_n(0) = A_n$, $\gamma_n(1) = B_n$, and

$$\begin{aligned} &\lim_{n\to\infty} \dot{x}_n(t) = \dot{x}(t), \\ &\lim_{n\to\infty} \dot{y}_n(t) = \dot{y}(t) \end{aligned} \tag{2.65}$$

with uniform convergence in $t$ running over $[0,1]$. Now

$$u(B_n) - u(A_n) = \int_0^1 \Big[u_x\big(x_n(t), y_n(t)\big)\dot{x}_n(t) + u_y\big(x_n(t), y_n(t)\big)\dot{y}_n(t)\Big]\, dt,$$

so by (2.53), (2.64), (2.65), and Lebesgue's dominated convergence theorem,

$$\bar{u}(B) - \bar{u}(A) = \int_0^1 \big[r_1\dot{x}(t) + r_2\dot{y}(t)\big]\, dt. \tag{2.66}$$

Similarly,

$$\bar{v}(B) - \bar{v}(A) = \int_0^1 \big[l_1\dot{x}(t) + l_2\dot{y}(t)\big]\, dt. \tag{2.67}$$

Combining (2.66), (2.67) and the fact that $r_1 = l_1$ and $r_2 = l_2$, we get

$$h(A) = \bar{v}(A) - \bar{u}(A) = \bar{v}(B) - \bar{u}(B) = h(B).$$

This establishes the claim.

Now it is clear that with $c = \bar{u} - \bar{v}$ the function $z$ is continuous. Since $u$ and $v$ are of class $C^1$ over $D_1$ and $D_2$, respectively, then to prove that $z$ is of class $C^1$, it suffices to prove that $z$ is of class $C^1$ on $\Gamma$. This, however, immediately follows from (2.54) and the continuity of the function $z$. $\square$

The above theorem shows that, generically, bifurcation in the class of $C^1$ functions of the two solutions to (2.6) along a smooth curve $\Gamma$ (where $\Lambda \equiv 0$) is feasible. The next two corollaries will additionally give necessary and sufficient conditions for a bifurcation to occur in the set of functions of class $C^2$. Moreover, Corollary 2.31 shows that in the case when $\Omega$ is divided by a smooth curve $\Gamma$ along which $\Lambda$ vanishes the number of global solutions of class $C^2$ to (2.13) cannot be three.

**Corollary 2.31.** *Let $p = (0, 0, -1)$ and let $q = (q_1, q_2, q_3)$ be such that $q_1^2 + q_2^2 > 0$ and $\|q\| = 1$. For a given pair of continuous functions $E_1$ and $E_2$ defining system (2.13), suppose that the set $\{(x, y) \in \Omega : \Lambda = 0\}$ is a smooth curve $\Gamma$ such that $\Omega \setminus \Gamma = D_1 \cup D_2$, where $D_1$ and $D_2$ are disjoint open subsets of $\Omega$. Suppose that there exist two different solutions $u$ and $\hat{u}$ of class $C^2$ to (2.13) over $D_1$, and two different solutions $v$ and $\hat{v}$ of class $C^2$ to (2.13) over $D_2$, respectively, such that the pair $(\hat{u}, \hat{v})$ satisfies (2.53). Let*

$$g(x, y) = \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u(x', y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v(x', y').$$

*Suppose, moreover, that, for some constant $c_1$, the function*

$$z(x, y) = \begin{cases} u(x, y) & \text{if } (x, y) \in D_1, \\ g(x, y) & \text{if } (x, y) \in \Gamma, \\ v(x, y) + c_1 & \text{if } (x, y) \in D_2 \end{cases}$$

*is of class $C^2$ over $\Omega$. Let*

$$h(x, y) = \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} \hat{u}(x', y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} \hat{v}(x', y').$$

Then, for some constant $c_2$, the function

$$\hat{z}(x,y) = \begin{cases} \hat{u}(x,y) & \text{if } (x,y) \in D_1, \\ h(x,y) & \text{if } (x,y) \in \Gamma, \\ \hat{v}(x,y) + c_2 & \text{if } (x,y) \in D_2 \end{cases}$$

is of class $C^2$ over $\Omega$.

**Proof.** By Corollary 2.30, it is enough to show that the function $\hat{z}$ has continuous second-order partial derivatives over $\Omega$. As functions $\hat{u}$ and $\hat{v}$ are of class $C^2$ over $D_1$ and $D_2$, respectively, it suffices to show the existence and continuity of second-order partial derivatives of the function $\hat{z}$ on $\Gamma$. Let $(x,y) \in \Gamma$ be an arbitrarily fixed point. Since $u$ and $v$ bifurcate along $\Gamma$ in $C^2$ class, we get the following:

$$\lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{xx}(x',y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{xx}(x',y') = z_{xx}(x,y) = a,$$

$$\lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{yy}(x',y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{yy}(x',y') = z_{yy}(x,y) = b,$$

$$\lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{xy}(x',y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{xy}(x',y') = z_{xy}(x,y) = c. \tag{2.68}$$

By Lemma 2.6, for each $(x',y') \in D_1$,

$$\hat{u}_x(x',y') = \frac{2q_1q_2u_y(x',y') - (q_2^2 - q_1^2)u_x(x',y')}{q_1^2 + q_2^2}$$

and

$$\hat{u}_y(x',y') = \frac{2q_1q_2u_x(x',y') - (q_1^2 - q_2^2)u_y(x',y')}{q_1^2 + q_2^2}.$$

Similarly, for each $(x',y') \in D_2$,

$$\hat{v}_x(x',y') = \frac{2q_1q_2v_y(x',y') - (q_2^2 - q_1^2)v_x(x',y')}{q_1^2 + q_2^2}$$

and

$$\hat{v}_y(x',y') = \frac{2q_1q_2v_x(x',y') - (q_1^2 - q_2^2)v_y(x',y')}{q_1^2 + q_2^2}.$$

Hence, for each $(x', y') \in D_1$,

$$\hat{u}_{xx}(x', y') = \frac{2q_1q_2u_{yx}(x', y') - (q_2^2 - q_1^2)u_{xx}(x', y')}{q_1^2 + q_2^2},$$

$$\hat{u}_{yy}(x', y') = \frac{2q_1q_2u_{xy}(x', y') - (q_1^2 - q_2^2)u_{yy}(x', y')}{q_1^2 + q_2^2}, \tag{2.69}$$

$$\hat{u}_{xy}(x', y') = \frac{2q_1q_2u_{yy}(x', y') - (q_2^2 - q_1^2)u_{xy}(x', y')}{q_1^2 + q_2^2}.$$

Analogously, for each $(x', y') \in D_2$,

$$\hat{v}_{xx}(x', y') = \frac{2q_1q_2v_{yx}(x', y') - (q_2^2 - q_1^2)v_{xx}(x', y')}{q_1^2 + q_2^2},$$

$$\hat{v}_{yy}(x', y') = \frac{2q_1q_2v_{xy}(x', y') - (q_1^2 - q_2^2)v_{yy}(x', y')}{q_1^2 + q_2^2}, \tag{2.70}$$

$$\hat{v}_{xy}(x', y') = \frac{2q_1q_2v_{yy}(x', y') - (q_2^2 - q_1^2)v_{xy}(x', y')}{q_1^2 + q_2^2}.$$

Taking into account (2.68), (2.69), (2.70), and the fact that the mixed derivatives of functions $u$ and $\hat{u}$ over $D_1$, and $v$ and $\hat{v}$ over $D_2$ coincide, we get

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} \hat{u}_{xx}(x', y') = \frac{2q_1q_2c - (q_2^2 - q_1^2)a}{q_1^2 + q_2^2} = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} \hat{v}_{xx}(x', y'),$$

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} \hat{u}_{yy}(x', y') = \frac{2q_1q_2c - (q_1^2 - q_2^2)b}{q_1^2 + q_2^2} = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} \hat{v}_{yy}(x', y'),$$

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} \hat{u}_{xx}(x', y') = \frac{2q_1q_2b - (q_1^2 - q_2^2)c}{q_1^2 + q_2^2} = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} \hat{v}_{xx}(x', y').$$

Hence, $\hat{u}$ and $\hat{v}$ bifurcate along $\Gamma$ in $C^2$ class and the function $\hat{z}$ is of class $C^2$. □

We now establish the following:

**Corollary 2.32.** *Let* $p = (0, 0, -1)$ *and let* $q = (q_1, q_2, q_3)$ *be such that* $q_1^2 + q_2^2 > 0$ *and* $\|q\| = 1$. *For a given pair of continuous functions* $E_1$ *and* $E_2$ *defining system*

(2.13), *suppose that the set* $\{(x,y) \in \Omega : \Lambda = 0\}$ *is a smooth curve* $\Gamma$ *such that* $\Omega \setminus \Gamma = D_1 \cup D_2$, *where* $D_1$ *and* $D_2$ *are disjoint open subsets of* $\Omega$. *Assume that there exist two different solutions of class* $C^2$ *to* (2.13) *over* $D_1$ *and two different solutions of class* $C^2$ *to* (2.13) *over* $D_2$, *respectively. Let* $u$ *be a solution over* $D_1$ *and* $v$ *be a solution over* $D_2$ *such that*

$$h(x,y) = \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u(x',y') = \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v(x',y') + c,$$

*for some choice of constant* $c$. *Assume, moreover, that* $u$ *and* $v$ *satisfy* (2.53). *If* $q_1^2 - q_2^2 \neq 0$, *then the function*

$$z(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in D_1, \\ h(x,y) & \text{if } (x,y) \in \Gamma, \\ v(x,y) + c & \text{if } (x,y) \in D_2 \end{cases}$$

*is of class* $C^2$ *over* $\Omega$ *if and only if, for each* $(x,y) \in \Gamma$,

$$\begin{aligned} \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{xx}(x',y') &= \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{xx}(x',y'), \\ \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{yy}(x',y') &= \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{yy}(x',y'). \end{aligned} \tag{2.71}$$

*If* $q_1^2 - q_2^2 = 0$, *then the function* $z$ *is of class* $C^2$ *over* $\Omega$ *if and only if, for each* $(x,y) \in \Gamma$, *either*

$$\begin{aligned} \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{xx}(x',y') &= \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{xx}(x',y'), \\ \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{xy}(x',y') &= \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{xy}(x',y'), \end{aligned} \tag{2.72}$$

*or*

$$\begin{aligned} \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{yy}(x',y') &= \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{yy}(x',y'), \\ \lim_{(x',y') \in D_1 \to (x,y) \in \Gamma} u_{xy}(x',y') &= \lim_{(x',y') \in D_2 \to (x,y) \in \Gamma} v_{xy}(x',y'). \end{aligned} \tag{2.73}$$

**Proof.** By Corollary 2.30, it is enough to show that the function $z$ has continuous second-order partial derivatives over $\Omega$. Since $u$ and $v$ are functions of class $C^2$

over $D_1$ and $D_2$, respectively, it suffices to show the existence and continuity of second-order partial derivatives of $z$ on $\Gamma$.

Necessity is immediate. To show sufficiency, consider first the case where $q_1^2 - q_2^2 \neq 0$. Let $(x, y)$ be a point in $\Gamma$. Taking into account (2.71), let

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} u_{xx}(x', y') = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} v_{xx}(x', y') = a,$$

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} u_{yy}(x', y') = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} v_{yy}(x', y') = b.$$

In view of the fact that the mixed derivatives of $u$ over $D_1$ and $v$ over $D_2$ coincide, it is enough to show that

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} u_{xy}(x', y') = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} v_{xy}(x', y'). \tag{2.74}$$

Since both $u$ and $v$ satisfy (2.14), it follows that for each $(x', y') \in D_1$

$$\frac{q_1 q_2}{q_1^2 - q_2^2}(u_{xx} - u_{yy}) = u_{xy}$$

and for each $(x', y') \in D_2$

$$\frac{q_1 q_2}{q_1^2 - q_2^2}(v_{xx} - u_{yy}) = v_{xy}.$$

Hence

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} u_{xy}(x', y') = \frac{q_1 q_2}{q_1^2 - q_2^2}(a - b) = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} v_{xy}(x', y').$$

Thus we get (2.74) which implies that the functions $u$ and $v$ bifurcate along $\Gamma$ in class $C^2$.

Assume now that $q_1^2 - q_2^2 = 0$ and, moreover, assume the validity of (2.72). Let $(x, y)$ be a point in $\Gamma$ and let

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} u_{xx}(x', y') = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} v_{xx}(x', y') = a,$$

$$\lim_{(x',y')\in D_1 \to (x,y)\in\Gamma} u_{xy}(x', y') = \lim_{(x',y')\in D_2 \to (x,y)\in\Gamma} v_{xy}(x', y') = b.$$

By the previous argument, it is enough to show that

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{yy}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{yy}(x',y'). \tag{2.75}$$

Since both $u$ and $v$ satisfy (2.14), we now have that for each $(x',y')\in D_1$

$$u_{xx} = u_{yy}$$

and for each $(x',y')\in D_2$

$$v_{xx} = v_{yy}.$$

Finally, (2.75) follows upon noting that

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{yy}(x',y') = a = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{yy}(x',y').$$

Thus $u$ and $v$ bifurcate along $\Gamma$ in class $C^2$.

The case when (2.73) holds is treated analogously. □

We now formulate a proposition whose proof will be omitted as it is similar to that of Corollary 2.32.

**Proposition 2.33.** *Let $p = (0,0,-1)$ and let $q = (q_1,q_2,q_3)$ be such that $q_1^2 + q_2^2 > 0$ and $\|q\| = 1$. For a given pair of continuous functions $E_1$ and $E_2$ defining system (2.13), suppose that the set $\{(x,y)\in\Omega : \Lambda = 0\}$ is a smooth curve $\Gamma$ such that $\Omega\setminus\Gamma = D_1\cup D_2$, where $D_1$ and $D_2$ are disjoint open subsets of $\Omega$. Assume that there exist two different solutions of class $C^2$ to (2.13) over $D_1$ and two different solutions of class $C^2$ to (2.13) over $D_2$, respectively. Let $u$ be a solution over $D_1$ and $v$ be a solution over $D_2$ such that*

$$h(x,y) = \lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v(x',y') + c,$$

*for some constant $c$. Assume, moreover, that $u$ and $v$ satisfy (2.53). If $q_1q_2\neq 0$, then the function*

$$z(x,y) = \begin{cases} u(x,y) & \text{if } (x,y)\in D_1, \\ h(x,y) & \text{if } (x,y)\in\Gamma, \\ v(x,y)+c & \text{if } (x,y)\in D_2 \end{cases}$$

*is of class $C^2$ over $\Omega$ if and only if, for each $(x,y)\in\Gamma$, either*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{xx}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{xx}(x',y'),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{xy}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{xy}(x',y'),$$

*or*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{yy}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{yy}(x',y'),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{xy}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{xy}(x',y').$$

*If $q_1q_2 = 0$, then the function $z$ is of class $C^2$ over $\Omega$ if and only if, for each $(x,y)\in\Gamma$,*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{xx}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{xx}(x',y'),$$

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma} u_{yy}(x',y') = \lim_{(x',y')\in D_2\to(x,y)\in\Gamma} v_{yy}(x',y').$$

The next and last corollary formulates neccessary and sufficient conditions for two solutions of class $C^2$ to (2.13), defined over $D_1$ and $D_2$, respectively, with graphs having zero Gaussian curvatures to bifurcate along $\Gamma$ in $C^2$ class and satisfy one of the representation conditions (2.25), (2.26), (2.27), or (2.28) from Proposition 2.13.

**Corollary 2.34.** *Let $p=(0,0,-1)$ and let $q=(q_1,q_2,q_3)$ be such that $q_1^2+q_2^2>0$ and $\|q\|=1$. For a given pair of continuous functions $E_1$ and $E_2$ defining the system (2.13), suppose that the set $\{(x,y)\in\Omega : \Lambda = 0\}$ is a smooth curve $\Gamma$. Suppose, moreover, that $\Omega\setminus\Gamma = D_1\cup D_2$, where $D_1$ and $D_2$ are disjoint open subsets of $\Omega$. Assume that there exist exactly two solutions of class $C^2$ over $D_1$ and exactly two solutions of class $C^2$ over $D_2$ to (2.13) such that the graph of any of those solutions has zero Gaussian curvature. Then, retaining the notation from Proposition 2.13, if $q_1q_2=0$, then the pair*

$$u(x,y) = ax+b+\psi(y), \qquad v(x,y) = a_1x+b_1+\psi_1(y)$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\psi''(y)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\psi_1''(y);$$

*the pair*

$$u(x,y)=\phi(x)+cy+d,\qquad v(x,y)=a_1x+b_1+\psi_1(y)$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\phi''(x)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\psi_1''(y)=0;$$

*the pair*

$$u(x,y)=\phi(x)+cx+d,\qquad v(x,y)=\phi_1(x)+c_1+d_1$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\phi''(x)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\phi_1''(x);$$

*and the pair*

$$u(x,y)=ax+b+\psi(y),\qquad v(x,y)=\phi_1(x)+c_1y+d_1$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\psi''(y)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\phi_1''(x)=0.$$

*If, in turn,* $q_1q_2\neq 0$*, then the pair*

$$u(x,y)=a(q_1x+q_2y)+b+\psi(q_1y-q_2x),\quad v(x,y)=a_1(q_1x+q_2y)+b_1+\psi_1(q_1y-q_2x)$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\psi''(q_1y-q_2x)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\psi_1''(q_1y-q_2x);$$

*the pair*

$$u(x,y)=\phi(q_1x+q_2y)+c(q_1y-q_2x)+d,\quad v(x,y)=a_1(q_1x+q_2y)+b_1+\psi_1(q_1y-q_2x)$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\phi''(q_1x+q_2y)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\psi_1''(q_1y-q_2x)=0;$$

*the pair*

$$u(x,y)=\phi(q_1x+q_2y)+c(q_1y-q_2x)+d,\quad v(x,y)=\phi_1(q_1x+q_2y)+c_1(q_1y-q_2x)+d_1$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\phi''(x)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\phi_1''(x);$$

*and the pair*

$$u(x,y)=a(q_1x+q_2y)+b+\psi(q_1y-q_2x),\quad v(x,y)=\phi_1(q_1x+q_2y)+c_1(q_1y-q_2x)+d_1$$

*bifurcates along* $\Gamma$ *in* $C^2$ *class if and only if*

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}=\psi''(q_1y-q_2x)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\phi_1''(q_1x+q_2y)=0.$$

The proof is elementary and as such is omitted. Note that, in view of Corollary 2.30, the constants $a, b, c, d, a_1, b_1, c_1$, and $d_1$ are chosen in such a way that each pair $(u, v)$ bifurcates along $\Gamma$ in $C^1$ class.

The last two corollaries can be illustrated by the following example.

**Example 2.35.**

($i$) The functions $v_{12}$ and $v_{21}$ from Example 2.27 are not of $C^2$ class along $\Gamma$ since neither condition (2.72) nor condition (2.73) from Corollary 2.32 are satisfied. However, the functions $v_{11}$ and $v_{22}$ are of class $C^2$ along $\Gamma$ since (2.72) and (2.73) are satisfied.

($ii$) The functions $v_{11}$, $v_{12}$, $v_{21}$, and $v_{22}$ from Example 2.26 are all of class $C^2$ along $\Gamma$ as the sufficient condition

$$\lim_{(x',y')\in D_1\to(x,y)\in\Gamma}\phi''(x)=\lim_{(x',y')\in D_2\to(x,y)\in\Gamma}\phi_1''(x),$$

from Corollary 2.34, is satisfied.

## 2.4. Impossible shading in two-source photometric stereo

Finally, we discuss the issue of *impossible shading* for two-source photometric stereo. To determine such shading, we require a combination of $E_1$, $E_2$, $p$, and $q$ for which there is no solution to (2.1) corresponding to a genuine shape, that is, a shape of class $C^2$ possessing a continuous extension over the boundary $\partial\Omega$. The last constraint implies that $u$ is bounded over $\Omega$ (we obviously assume that the closure of $\Omega$ is compact). In the last example we reveal different quadruplets $E_1$, $E_2$, $p$, $q$ yielding impossible shading.

**Example 2.36.**

(*i*) Assume that at least one of the functions $(E_1, E_2)$ takes at least one value greater than 1. Then by the Cauchy-Schwartz-Buniakowski inequality there is no function $u$ of class $C^1$ satisfying equation (2.1).

(*ii*) Assume that continuous functions $E_1$, $E_2$ and linearly independent vectors $p$, $q$ generate the system (2.1) over some region $\Omega$ in such a way that either $\Lambda$, defined by (2.3), is negative at least at one point in $\Omega$, or both $\sigma^+$ and $\sigma^-$, introduced in Corollary 2.3, are negative at least at one point in $\Omega$. Then, by Theorem 2.1, there is no algebraic solution $(u_x, u_y)$ to the system (2.1). In particular, there is no function $u$ satisfying (2.1). Hence, the above brightness patterns $(E_1, E_2)$ could not be the images of any smooth Lambertian surface.

(*iii*) Let $p = (0,0,-1)$ and $q = (0,1,-1)$ and let $E_1(x,y) = x(1+x^2)^{-1/2}$ and $E_2(x,y) = x\big[2(1+x^2)\big]^{-1/2}$. Consider the corresponding image irradiance equations

$$\frac{1}{\sqrt{u_x^2+u_y^2+1}} = \frac{x}{\sqrt{x^2+1}},$$
$$\frac{1+u_y}{\sqrt{2}\sqrt{u_x^2+u_y^2+1}} = \frac{x}{\sqrt{2}\sqrt{x^2+1}} \tag{2.76}$$

over any simply connected $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\}$. Clearly,

$$\Lambda = \frac{1}{x^2+1} > 0.$$

By Corollary 2.2, there exist at most two solutions $u^+$ and $u^-$ over $\Omega$ satisfying system (2.76). An easy verification shows that

$$u^+(x,y) = \ln x \quad \text{and} \quad u^-(x,y) = -\ln x.$$

Obviously both these functions are unbounded over $\Omega$ and as such do not correspond to any physically realisable shapes. Note that if vector $p$ is overhead, $\|p\| = \|q\| = 1$, and $\Lambda > 0$, then Theorem 2.10 assures that either both solutions are bounded or both are unbounded. Such a dichotomy occurs neither in the case of a single-overhead-source problem (see [3]), nor in the case when $\Lambda \geq 0$ in two-source photometric stereo (an easy modification of Example 2.29 provides a desired counter-example).

(*iv*) Assume that the vectors $p$ and $q$ are as in Example 2.29. Consider the functions

$$u_1(x,y) = \sin^3\frac{1}{x} \qquad \text{and} \qquad u_2(x,y) = -\sin^3\frac{1}{x}$$

over $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1/\pi\}$. By repeating the construction from Example 2.29, we can only generate, over $\Omega$, unbounded solutions or bounded solutions admitting no continuous extension on the boundary. Of course, any such solution cannot represent a physically realisable surface. Moreover, we get an example in which there are bounded and unbounded solutions to a two-source photometric stereo problem.

(*v*) Assume that $p = (0,0,-1)$ and $q = (0,1,-1)$. Consider the pair of continuous images

$$E_1(x,y) = \begin{cases} \frac{1}{\sqrt{x^2+1}} & \text{if } x \geq 0, \\ \\ \frac{1}{\sqrt{x^4+1}} & \text{if } x < 0, \end{cases}$$

$$E_2(x,y) = \begin{cases} \frac{1}{\sqrt{2}\sqrt{x^2+1}} & \text{if } x \geq 0, \\ \\ \frac{1}{\sqrt{2}\sqrt{x^4+1}} & \text{if } x < 0, \end{cases}$$

and the corresponding system of image irradiance equations

$$\frac{1}{\sqrt{u_x^2+u_y^2+1}} = \begin{cases} \frac{1}{\sqrt{x^2+1}} & \text{if } x \geq 0, \\ \frac{1}{\sqrt{x^4+1}} & \text{if } x < 0, \end{cases}$$

$$\frac{u_y+1}{\sqrt{2}\sqrt{u_x^2+u_y^2+1}} = \begin{cases} \frac{1}{\sqrt{2}\sqrt{x^2+1}} & \text{if } x \geq 0, \\ \frac{1}{\sqrt{2}\sqrt{x^4+1}} & \text{if } x < 0. \end{cases} \tag{2.77}$$

It is easy to verify that

$$\Lambda = \begin{cases} 1-\frac{1}{x^2+1} & \text{if } x \geq 0, \\ 1-\frac{1}{x^4+1} & \text{if } x < 0. \end{cases}$$

Hence, $\Lambda = 0$ if and only if $x = 0$. By Corollaries 2.2 and 2.3, there exist at most two solutions $u^+$ and $u^-$ of class $C^2$ to (2.77) over any region $D_1$ contained in the semiplane with $x > 0$. An easy verification shows that

$$u^+(x,y) = \frac{x^2}{2} \qquad \text{and} \qquad u^-(x,y) = -\frac{x^2}{2}.$$

Similarly, for any $D_2$ contained in the semiplane with $x < 0$, there are at most two solutions $v^+$ and $v^-$ to (2.77) of class $C^2$ over $D_2$. A simple calculation yields

$$v^+(x,y) = \frac{x^3}{3} \qquad \text{and} \qquad v^-(x,y) = -\frac{x^3}{3}.$$

Now, if we are looking for any solution of class $C^2$ over any $\Omega$ containing the $y$-axis, a possible bifurcation along the $y$-axis assures that there are only four candidate solutions:

$$u_1(x,y) = \begin{cases} \frac{x^2}{2} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ \frac{x^3}{3} & \text{if } x < 0, \end{cases}$$

$$u_2(x,y) = \begin{cases} \frac{x^2}{2} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\frac{x^3}{3} & \text{if } x < 0, \end{cases}$$

$$u_3(x,y) = \begin{cases} -\frac{x^2}{2} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ \frac{x^3}{3} & \text{if } x < 0, \end{cases}$$

$$u_4(x,y) = \begin{cases} -\frac{x^2}{2} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\frac{x^3}{3} & \text{if } x < 0. \end{cases}$$

None of them is of class $C^2$ along the $y$-axis. This can be verified by direct calculation or can be inferred from Corollary 2.34. Hence there is no smooth integrable surface satisfying (2.77) over $\Omega$.

*(vi)* Let $p = (1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$, $q = (-1/\sqrt{6}, -1/\sqrt{6}, -\sqrt{2/3})$, $E_1(x,y) = \sin(x^2 + y^2)$, and $E_2(x,y) = \cos(x^2 + y^2)$. Consider the corresponding system of two image irradiance equations

$$\begin{aligned} \frac{u_x + u_y + 1}{\sqrt{3}\sqrt{u_x^2 + u_y^2 + 1}} &= \sin(x^2 + y^2), \\ \frac{-u_x - u_y + 2}{\sqrt{6}\sqrt{u_x^2 + u_y^2 + 1}} &= \cos(x^2 + y^2) \end{aligned} \tag{2.78}$$

over $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \pi/2\}$. It is easy to verify that

$$\Lambda = 1 - \sin^2(x^2 + y^2) - \cos^2(x^2 + y^2) = 0.$$

By Corollary 2.16, there exists at most one solution $u$ to (2.78), which, by (2.24), satisfies

$$u_x = -\frac{p_1 E_1 + q_1 E_2}{p_3 E_1 + q_3 E_2} = \frac{\sqrt{2}\sin(x^2 + y^2) - \cos(x^2 + y^2)}{\sqrt{2}\sin(x^2 + y^2) + 2\cos(x^2 + y^2)},$$

$$u_y = -\frac{p_2 E_1 + q_2 E_2}{p_3 E_1 + q_3 E_2} = \frac{\sqrt{2}\sin(x^2 + y^2) - \cos(x^2 + y^2)}{\sqrt{2}\sin(x^2 + y^2) + 2\cos(x^2 + y^2)}.$$

An easy verification shows that the integrability condition $(u_x)_y = (u_y)_x$ is not fulfilled over any open subregion $\Omega$ of $\mathbb{R}^2$. Hence, by Corollary 2.17, there is no function $u$ of class $C^2$ whose first-order partial derivatives satisfy (2.78) over $\Omega$.

# Appendix 1

In this appendix we will prove the following:

**Lemma 2.5.** *Suppose that* $p = (p_1, p_2, p_3)$ *and* $q = (q_1, q_2, q_3)$ *satisfy* $\|p\| = \|q\| = 1$. *Let* $u$ *be a solution to* (2.1). *Then*

$$\Lambda = \frac{\big[u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1) + p_2q_1 - p_1q_2\big]^2}{u_x^2 + u_y^2 + 1},$$

*where*

$$\Lambda(x,y) = 1 - E_1^2(x,y) - E_2^2(x,y) - \langle p|q\rangle\big(\langle p|q\rangle - 2E_1(x,y)E_2(x,y)\big),$$

$$E_1(x,y) = \frac{p_1u_x + p_2u_y - p_3}{\sqrt{u_x^2 + u_y^2 + 1}}, \quad \text{and} \quad E_2(x,y) = \frac{q_1u_x + q_2u_y - q_3}{\sqrt{u_x^2 + u_y^2 + 1}}.$$

**Proof.** Letting $n = (u_x, u_y, -1)$, we have

$$\begin{aligned}
\|n\|^2\Lambda &= \|n\|^2 + 2\langle p|q\rangle\langle p|n\rangle\langle q|n\rangle - \langle p|n\rangle^2 - \langle q|n\rangle^2 - \langle p|q\rangle\|n\|^2 \\
&= u_x^2 + u_y^2 + 2(p_1q_1 + p_2q_2 + p_3q_3)(p_1u_x + p_2u_y - p_3)(q_1u_x + q_2u_y - q_3) \\
&\quad - (p_1u_x + p_2u_y - p_3)^2 - (q_1u_x + q_2u_y - q_3)^2 + 1 \\
&\quad - (p_1q_1 + p_2q_2 + p_3q_3)^2(u_x^2 + u_y^2 + 1) \\
&= u_x^2 + u_y^2 + 1 + 2(p_1^2q_1u_x + p_1p_2q_1u_y - p_1p_3q_1 + p_1p_2q_2u_x + p_2^2q_2u_y \\
&\qquad - p_2p_3q_2 + p_1p_3q_3u_x + p_2p_3q_3u_y - p_3^2q_3)(q_1u_x + q_2u_y - q_3) \\
&\quad - \big[p_1^2u_x^2 + p_2^2u_y^2 + p_3^2 + 2p_1p_2u_xu_y - 2p_3(p_1u_x + p_2u_y)\big] \\
&\quad - \big[q_1^2u_x^2 + q_2^2u_y^2 + q_3^2 + 2q_1q_2u_xu_y - 2q_3(q_1u_x + q_2u_y)\big] \\
&\quad - \big[p_1^2q_1^2 + p_2^2q_2^2 + p_3^2q_3^2 + 2p_1p_2q_1q_2 + 2p_3q_3(p_1q_1 + p_2q_2)\big](u_x^2 + u_y^2 + 1)
\end{aligned}$$

$$
\begin{aligned}
&= u_x^2 + u_y^2 + 1 + 2(p_1^2 q_1^2 u_x^2 + p_1 p_2 q_1^2 u_x u_y - p_1 p_3 q_1^2 u_x + p_1 p_2 q_1 q_2 u_x^2 \\
&\quad + p_2^2 q_1 q_2 u_x u_y - p_2 p_3 q_1 q_2 u_x + p_1 p_3 q_1 q_3 u_x^2 + p_2 p_3 q_1 q_3 u_x u_y - p_3^2 q_1 q_3 u_x) \\
&\quad + 2(p_1^2 q_1 q_2 u_x u_y + p_1 p_2 q_1 q_2 u_y^2 - p_1 p_3 q_1 q_2 u_y + p_1 p_2 q_2^2 u_x u_y + p_2^2 q_2^2 u_y^2 \\
&\quad - p_2 p_3 q_2^2 u_y + p_1 p_3 q_2 q_3 u_x u_y + p_2 p_3 q_2 q_3 u_y^2 - p_3^2 q_2 q_3 u_y) \\
&\quad + 2(-p_1^2 q_1 q_3 u_x - p_1 p_2 q_1 q_3 u_y + p_1 p_3 q_1 q_3 - p_1 p_2 q_2 q_3 u_x - p_2^2 q_2 q_3 u_y \\
&\quad + p_2 p_3 q_2 q_3 - p_1 p_3 q_3^2 u_x - p_2 p_3 q_3^2 u_y + p_3^2 q_3^2) \\
&\quad - p_1^2 u_x^2 - p_2^2 u_y^2 - p_3^2 - 2 p_1 p_2 u_x u_y + 2 p_1 p_3 u_x + 2 p_2 p_3 u_y \\
&\quad - q_1^2 u_x^2 - q_2^2 u_y^2 - q_3^2 - 2 q_1 q_2 u_x u_y + 2 q_1 q_3 u_x + 2 q_2 q_3 u_y \\
&\quad - p_1^2 q_1^2 u_x^2 - p_2^2 q_2^2 u_x^2 - p_3^2 q_3^2 u_x^2 - 2 p_1 p_2 q_1 q_2 u_x^2 - 2 p_1 p_3 q_1 q_3 u_x^2 - 2 p_2 p_3 q_2 q_3 u_x^2 \\
&\quad - p_1^2 q_1^2 u_y^2 - p_2^2 q_2^2 u_y^2 - p_3^2 q_3^2 u_y^2 - 2 p_1 p_2 q_1 q_2 u_y^2 - 2 p_1 p_3 q_1 q_3 u_y^2 - 2 p_2 p_3 q_2 q_3 u_y^2 \\
&\quad - p_1^2 q_1^2 - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_1 p_2 q_1 q_2 - 2 p_1 p_3 q_1 q_3 - 2 p_2 p_3 q_2 q_3.
\end{aligned}
$$

It is clear that

$$
\|n\|^2 \Lambda = \alpha u_x^2 + \beta u_y^2 + 2\gamma u_x u_y + 2\delta u_x + 2\varepsilon u_y + \zeta,
$$

where

$$
\begin{aligned}
\alpha &= 2 p_1^2 q_1^2 + 2 p_1 p_2 q_1 q_2 + 2 p_1 p_3 q_1 q_3 - p_1^2 - q_1^2 - p_1^2 q_1^2 - p_2^2 q_2^2 \\
&\quad - p_3^2 q_3^2 - 2 p_1 p_2 q_1 q_2 - 2 p_1 p_3 q_1 q_3 - 2 p_2 p_3 q_2 q_3 + 1 \\
&= - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_2 p_3 q_2 q_3 - p_1^2 - q_1^2 + p_1^2 q_1^2 + 1 \\
&= - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_2 p_3 q_2 q_3 - (1 - q_2^2 - q_3^2) - p_1^2 (1 - q_1^2) + 1 \\
&= - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_2 p_3 q_2 q_3 + q_2^2 + q_3^2 - p_1^2 (q_2^2 + q_3^2) \\
&= - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_2 p_3 q_2 q_3 + (q_2^2 + q_3^2)(1 - p_1^2) \\
&= - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_2 p_3 q_2 q_3 + (q_2^2 + q_3^2)(p_2^2 + p_3^2) \\
&= - p_2^2 q_2^2 - p_3^2 q_3^2 - 2 p_2 p_3 q_2 q_3 + p_2^2 q_2^2 + p_3^2 q_2^2 + p_2^2 q_3^2 + p_3^2 q_3^2 \\
&= p_3^2 q_2^2 + p_2^2 q_3^2 - 2 p_2 p_3 q_2 q_3 \\
&= (p_2 q_3 - p_3 q_2)^2,
\end{aligned}
$$

$$
\begin{aligned}
\beta &= 2p_2^2q_2^2 + 2p_1p_2q_1q_2 + 2p_2p_3q_2q_3 - p_2^2 - q_2^2 - p_1^2q_1^2 - p_2^2q_2^2 \\
&\quad - p_3^2q_3^2 - 2p_1p_2q_1q_2 - 2p_1p_3q_1q_3 - 2p_2p_3q_2q_3 + 1 \\
&= - p_1^2q_1^2 - p_3^2q_3^2 - 2p_1p_3q_1q_3 - p_2^2 - q_2^2 + p_2^2q_2^2 + 1 \\
&= - p_1^2q_1^2 - p_3^2q_3^2 - 2p_1p_3q_1q_3 - (1 - p_1^2 - p_3^2) - q_2^2(1 - p_2^2) + 1 \\
&= - p_1^2q_1^2 - p_3^2q_3^2 - 2p_1p_3q_1q_3 + p_1^2 + p_3^2 - q_2^2(p_1^2 + p_3^2) \\
&= - p_1^2q_1^2 - p_3^2q_3^2 - 2p_1p_3q_1q_3 + (p_1^2 + p_3^2)(1 - q_2^2) \\
&= - p_1^2q_1^2 - p_3^2q_3^2 - 2p_1p_3q_1q_3 + (p_1^2 + p_3^2)(q_1^2 + q_3^2) \\
&= - p_1^2q_1^2 - p_3^2q_3^2 - 2p_1p_3q_1q_3 + p_1^2q_1^2 + p_1^2q_3^2 + p_3^2q_1^2 + p_3^2q_3^2 \\
&= p_1^2q_3^2 + p_3^2q_1^2 - 2p_1p_3q_1q_3 \\
&= (p_1q_3 - p_3q_1)^2,
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= p_1p_2q_1^2 + p_2^2q_1q_2 + p_2p_3q_1q_3 + p_1^2q_1q_2 + p_1p_2q_2^2 + p_1p_3q_2q_3 - p_1p_2 - q_1q_2 \\
&= p_1p_2(1 - q_2^2 - q_3^2) + q_1q_2(1 - p_1^2 - p_3^2) + p_2p_3q_1q_3 + p_1^2q_1q_2 \\
&\quad + p_1p_2q_2^2 + p_1p_3q_2q_3 - p_1p_2 - q_1q_2 \\
&= p_1p_2 - p_1p_2q_2^2 - p_1p_2q_3^2 + q_1q_2 - p_1^2q_1q_2 - p_3^2q_1q_2 + p_2p_3q_1q_3 \\
&\quad + p_1^2q_1q_2 + p_1p_2q_2^2 + p_1p_3q_2q_3 - p_1p_2 - q_1q_2 \\
&= - p_1p_2q_3^2 - p_3^2q_1q_2 + p_2p_3q_1q_3 + p_1p_3q_2q_3 \\
&= p_3q_2(p_1q_3 - p_3q_1) + p_2q_3(p_3q_1 - p_1q_3) \\
&= (p_1q_3 - p_3q_1)(p_3q_2 - p_2q_3),
\end{aligned}
$$

$$
\begin{aligned}
\delta &= p_1p_3 + q_1q_3 - p_1p_3q_1^2 - p_2p_3q_1q_2 - p_3^2q_1q_3 - p_1^2q_1q_3 - p_1p_2q_2q_3 - p_1p_3q_3^2 \\
&= - p_1p_3(1 - q_2^2 - q_3^2) - q_1q_3(1 - p_1^2 - p_2^2) - p_2p_3q_1q_2 - p_1^2q_1q_3 \\
&\quad - p_1p_2q_2q_3 - p_1p_3q_3^2 + p_1p_3 + q_1q_3 \\
&= p_1p_3 + q_1q_3 - p_1p_3 + p_1p_3q_2^2 + p_1p_3q_3^2 - q_1q_3 + p_1^2q_1q_3 + p_2^2q_1q_3 \\
&\quad - p_2p_3q_1q_2 - p_1^2q_1q_3 - p_1p_2q_2q_3 - p_1p_3q_3^2
\end{aligned}
$$

$$
\begin{aligned}
&= p_1p_3q_2^2 + p_2^2q_1q_3 - p_2p_3q_1q_2 - p_1p_2q_2q_3 \\
&= p_3q_2(p_1q_2 - p_2q_1) + p_2q_3(p_2q_1 - p_1q_2) \\
&= (p_1q_2 - p_2q_1)(p_3q_2 - p_2q_3),
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon &= -p_1p_3q_1q_2 - p_2p_3q_2^2 - p_3^2q_2q_3 - p_1p_2q_1q_3 - p_2^2q_2q_3 - p_2p_3q_3^2 + p_2p_3 + q_2q_3 \\
&= -p_2p_3(1 - q_1^2 - q_3^2) - q_2q_3(1 - p_1^2 - p_3^2) - p_3^2q_2q_3 - p_1p_2q_1q_3 \\
&\quad - p_2p_3q_3^2 - p_1p_3q_1q_2 + p_2p_3 + q_2q_3 \\
&= p_2p_3 + q_2q_3 - p_2p_3 + p_2p_3q_1^2 + p_2p_3q_3^2 - q_2q_3 + p_1^2q_2q_3 + p_3^2q_2q_3 \\
&\quad - p_3^2q_2q_3 - p_1p_2q_1q_3 - p_2p_3q_3^2 - p_1p_3q_1q_2 \\
&= p_2p_3q_1^2 + p_1^2q_2q_3 - p_1p_2q_1q_3 - p_1p_3q_1q_2 \\
&= p_2q_1(p_3q_1 - p_1q_3) + p_1q_2(p_1q_3 - p_3q_1) \\
&= (p_1q_3 - p_3q_1)(p_1q_2 - p_2q_1),
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta &= 1 + p_1^2q_1^2 - p_2^2q_2^2 - p_3^2q_3^2 - 2p_1p_2q_1q_2 - 2p_1p_3q_1q_3 - 2p_2p_3q_2q_3 \\
&\quad + 2p_3^2q_3^2 + 2p_1p_3q_1q_3 + 2p_2p_3q_2q_3 - p_3^2 - q_3^2 \\
&= 1 - p_1^2q_1^2 - p_2^2q_2^2 - 2p_1p_2q_1q_2 - p_3^2 - q_3^2 + p_3^2q_3^2 \\
&= 1 - p_1^2q_1^2 - p_2^2q_2^2 - 2p_1p_2q_1q_2 - (1 - p_1^2 - p_2^2) - q_3^2(1 - p_3^2) \\
&= -p_1^2q_1^2 - p_2^2q_2^2 - 2p_1p_2q_1q_2 + p_1^2 + p_2^2 - q_3^2(p_1^2 + p_2^2) \\
&= -p_1^2q_1^2 - p_2^2q_2^2 - 2p_1p_2q_1q_2 + (p_1^2 + p_2^2)(1 - q_3^2) \\
&= -p_1^2q_1^2 - p_2^2q_2^2 - 2p_1p_2q_1q_2 + (p_1^2 + p_2^2)(q_1^2 + q_2^2) \\
&= -p_1^2q_1^2 - p_2^2q_2^2 - 2p_1p_2q_1q_2 + p_1^2q_1^2 + p_1^2q_2^2 + p_2^2q_1^2 + p_2^2q_2^2 \\
&= p_1^2q_2^2 + p_2^2q_1^2 - 2p_1p_2q_1q_2 \\
&= (p_1q_2 - p_2q_1)^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|n\|^2\Lambda &= u_x^2(p_2q_3 - p_3q_2)^2 + u_y^2(p_1q_3 - p_3q_1)^2 - 2u_xu_y(p_1q_3 - p_3q_1)(p_2q_3 - p_3q_2) \\
&\quad + 2u_x(p_1q_2 - p_2q_1)(p_3q_2 - p_2q_3) + 2u_y(p_1q_3 - p_3q_1)(p_1q_2 - p_2q_1) \\
&\quad + (p_1q_2 - p_2q_1)^2 \\
&= \big[u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1)\big]^2 + (p_1q_2 - p_2q_1)^2 \\
&\quad + 2u_x(p_1q_2 - p_2q_1)(p_3q_2 - p_2q_3) + 2u_y(p_3q_1 - p_1q_3)(p_2q_1 - p_1q_2) \\
&= \big[u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1)\big]^2 + (p_1q_2 - p_2q_1)^2 \\
&\quad + 2(p_1q_2 - p_2q_1)\big[u_x(p_3q_2 - p_2q_3) - u_y(p_3q_1 - p_1q_3)\big] \\
&= \big[u_x(p_3q_2 - p_2q_3) - u_y(p_3q_1 - p_1q_3) + p_1q_2 - p_2q_1\big]^2 \\
&= \big[u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1) + p_2q_1 - p_1q_2\big]^2. \quad \square
\end{aligned}
$$

# Appendix 2

This appendix is devoted to the proof of Lemma 2.6 which reads:

**Lemma 2.6.** *Suppose that* $p = (p_1, p_2, p_3)$ *and* $q = (q_1, q_2, q_3)$ *satisfy* $\|p\| = \|q\| = 1$. *Given a function* $u$ *of class* $C^1$ *over a domain* $\Omega$, *let* $E_1$ *and* $E_2$ *be defined by* (2.1). *Suppose that* $\Lambda$, *given by* (2.3), *is positive and, for each choice of sign,*

$$\sigma^{\pm} = (p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 \pm (p_1q_2 - p_2q_1)\sqrt{\Lambda}$$

*does not vanish over* $\Omega$. *Let*

$$\theta = u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1) + p_2q_1 - p_1q_2.$$

*Then*

$$\frac{(q_1\langle p|q\rangle - p_1)E_1 + (p_1\langle p|q\rangle - q_1)E_2 + (p_3q_2 - p_2q_3)\varepsilon\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\varepsilon\sqrt{\Lambda}}$$

$$= \begin{cases} u_x & \textit{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta > 0, \\[2ex] \dfrac{(a^2 - b^2 - c^2)u_x + 2acu_y + 2ab}{2abu_x + 2bcu_y + b^2 - a^2 - c^2} & \textit{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta < 0, \end{cases}$$

*and*

$$\frac{(q_2\langle p|q\rangle - p_2)E_1 + (p_2\langle p|q\rangle - q_2)E_2 + (p_1q_3 - p_3q_1)\varepsilon\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\varepsilon\sqrt{\Lambda}}$$

$$= \begin{cases} u_y & \textit{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta > 0, \\[2ex] \dfrac{2acu_x + (c^2 - a^2 - b^2)u_y + 2bc}{2abu_x + 2bcu_y + b^2 - a^2 - c^2} & \textit{if } \operatorname{sgn}\varepsilon \operatorname{sgn}\theta < 0, \end{cases}$$

*where*

$$a = p_3q_2 - p_2q_3,\ b = p_1q_2 - p_2q_1,\ \textit{and}\ c = p_1q_3 - p_3q_1,$$

*and function* $\varepsilon = \varepsilon(x, y)$ *is everywhere constant taking value* $\pm 1$.

**Proof.** Suppose that

$$\theta = u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1) + p_2q_1 - p_1q_2 > 0.$$

Assume, moreover, that $\varepsilon(x, y) = 1$. Then, obviously, $\operatorname{sgn}\varepsilon \operatorname{sgn}\theta > 0$. By Lemma 2.5 and (2.2),

$$\frac{(q_1\langle p|q\rangle - p_1)E_1 + (p_1\langle p|q\rangle - q_1)E_2 + (p_3q_2 - p_2q_3)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\sqrt{\Lambda}} = \frac{L_1}{L_2} \tag{A.2.1}$$

and

$$\frac{(q_2\langle p|q\rangle - p_2)E_1 + (p_2\langle p|q\rangle - q_2)E_2 + (p_1q_3 - p_3q_1)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\sqrt{\Lambda}} = \frac{L_1'}{L_2}, \tag{A.2.2}$$

where

$$\begin{aligned}
L_1 = {} & p_1^2q_1^2u_x + p_1p_2q_1q_2u_x + p_1p_3q_1q_3u_x - p_1^2u_x \\
& + p_1p_2q_1^2u_y + p_2^2q_1q_2u_y + p_2p_3q_1q_3u_y - p_1p_2u_y \\
& - p_1p_3q_1^2 - p_2p_3q_1q_2 - p_3^2q_1q_3 + p_1p_3 \\
& + p_1^2q_1^2u_x + p_1p_2q_1q_2u_x + p_1p_3q_1q_3u_x - q_1^2u_x \\
& + p_1^2q_1q_2u_y + p_1p_2q_2^2u_y + p_1p_3q_2q_3u_y - q_1q_2u_y \\
& - p_1^2q_1q_3 - p_1p_2q_2q_3 - p_1p_3q_3^2 + q_1q_3 \\
& + p_2p_3q_2q_3u_x - p_3^2q_2^2u_x - p_1p_3q_2q_3u_y + p_3^2q_1q_2u_y + p_2p_3q_1q_2 - p_1p_3q_2^2 \\
& - p_2^2q_3^2u_x + p_2p_3q_2q_3u_x + p_1p_2q_3^2u_y - p_2p_3q_1q_3u_y - p_2^2q_1q_3 + p_1p_2q_2q_3,
\end{aligned}$$

$$L_2 = p_1p_3u_x - p_1^2q_1q_3u_x - p_1p_2q_2q_3u_x - p_1p_3q_3^2u_x$$

$$
\begin{aligned}
&+ p_2p_3u_y - p_1p_2q_1q_3u_y - p_2^2q_2q_3u_y - p_2p_3q_3^2u_y \\
&- p_3^2 + p_1p_3q_1q_3 + p_2p_3q_2q_3 + p_3^2q_3^2 \\
&+ q_1q_3u_x - p_1p_3q_1^2u_x - p_2p_3q_1q_2u_x - p_3^2q_1q_3u_x \\
&+ q_2q_3u_y - p_1p_3q_1q_2u_y - p_2p_3q_2^2u_y - p_3^2q_2q_3u_y \\
&- q_3^2 + p_1p_3q_1q_3 + p_2p_3q_2q_3 + p_3^2q_3^2 \\
&+ p_1p_2q_2q_3u_x - p_1p_3q_2^2u_x - p_1^2q_2q_3u_y + p_1p_3q_1q_2u_y + p_1p_2q_1q_2 - p_1^2q_2^2 \\
&- p_2^2q_1q_3u_x + p_2p_3q_1q_2u_x + p_1p_2q_1q_3u_y - p_2p_3q_1^2u_y - p_2^2q_1^2 + p_1p_2q_1q_2,
\end{aligned}
$$

and

$$
\begin{aligned}
L_1' = {} & p_1^2q_1q_2u_x + p_1p_2q_2^2u_x + p_1p_3q_2q_3u_x - p_1p_2u_x \\
&+ p_1p_2q_1q_2u_y + p_2^2q_2^2u_y + p_2p_3q_2q_3u_y - p_2^2u_y \\
&- p_1p_3q_1q_2 - p_2p_3q_2^2 - p_3^2q_2q_3 + p_2p_3 \\
&+ p_1p_2q_1^2u_x + p_2^2q_1q_2u_x + p_2p_3q_1q_3u_x - q_1q_2u_x \\
&+ p_1p_2q_1q_2u_y + p_2^2q_2^2u_y + p_2p_3q_2q_3u_y - q_2^2u_y \\
&- p_1p_2q_1q_3 - p_2^2q_2q_3 - p_2p_3q_3^2 + q_2q_3 \\
&+ p_1p_2q_3^2u_x - p_1p_3q_2q_3u_x - p_1^2q_3^2u_y + p_1p_3q_1q_3u_y + p_1p_2q_1q_3 - p_1^2q_2q_3 \\
&- p_2p_3q_1q_3u_x + p_3^2q_1q_2u_x + p_1p_3q_1q_3u_y - p_3^2q_1^2u_y - p_2p_3q_1^2 + p_1p_3q_1q_2.
\end{aligned}
$$

It is plain that $L_1$ takes the form

$$
L_1 = A_1u_x + B_1u_y + C_1,
$$

where

$$
\begin{aligned}
A_1 = {} & p_1^2q_1^2 + p_1p_2q_1q_2 + p_1p_3q_1q_3 - p_1^2 + p_1^2q_1^2 + p_1p_2q_1q_2 \\
&+ p_1p_3q_1q_3 - q_1^2 + p_2p_3q_2q_3 - p_3^2q_2^2 - p_2^2q_3^2 + p_2p_3q_2q_3 \\
= {} & 2p_1p_2q_1q_2 + 2p_1p_3q_1q_3 + 2p_2p_3q_2q_3 + 2p_1^2q_1^2 - p_1^2 - q_1^2 \\
&- q_2^2(1 - p_1^2 - p_2^2) - p_2^2(1 - q_1^2 - q_2^2) \\
= {} & 2(p_1p_2q_1q_2 + p_1p_3q_1q_3 + p_2p_3q_2q_3) + 2(p_1^2q_1^2 + p_2^2q_2^2)
\end{aligned}
$$

$$- p_1^2 - p_2^2 - q_1^2 - q_2^2 + p_1^2q_2^2 + p_2^2q_1^2,$$

$$\begin{aligned}
B_1 &= p_1p_2q_1^2 + p_2^2q_1q_2 + p_2p_3q_1q_3 - p_1p_2 + p_1^2q_1q_2 + p_1p_2q_2^2 \\
&\quad + p_1p_3q_2q_3 - q_1q_2 - p_1p_3q_2q_3 + p_3^2q_1q_2 + p_1p_2q_3^2 - p_2p_3q_1q_3 \\
&= p_1p_2(q_1^2 + q_2^2 + q_3^2) + q_1q_2(p_1^2 + p_2^2 + p_3^2) - p_1p_2 - q_1q_2 \\
&= p_1p_2 + q_1q_2 - p_1p_2 - q_1q_2 \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
C_1 &= - p_1p_3q_1^2 - p_2p_3q_1q_2 - p_3^2q_1q_3 + p_1p_3 - p_1^2q_1q_3 - p_1p_2q_2q_3 \\
&\quad - p_1p_3q_3^2 + q_1q_3 + p_2p_3q_1q_2 - p_1p_3q_2^2 - p_2^2q_1q_3 + p_1p_2q_2q_3 \\
&= - p_1p_3(q_1^2 + q_2^2 + q_3^2) - q_1q_3(p_1^2 + p_2^2 + p_3^2) + p_1p_3 + q_1q_3 \\
&= p_1p_3 + q_1q_3 - p_1p_3 - q_1q_3 \\
&= 0.
\end{aligned}$$

Similarly, $L_2$ takes the form

$$L_2 = A_2u_x + B_2u_y + C_2,$$

where

$$\begin{aligned}
A_2 &= p_1p_3 - p_1^2q_1q_3 - p_1p_2q_2q_3 - p_1p_3q_3^2 + q_1q_3 - p_1p_3q_1^2 \\
&\quad - p_2p_3q_1q_2 - p_3^2q_1q_3 + p_1p_2q_2q_3 - p_1p_3q_2^2 - p_2^2q_1q_3 + p_2p_3q_1q_2 \\
&= p_1p_3 - p_1p_3(q_1^2 + q_2^2 + q_3^2) + q_1q_3 - q_1q_3(p_1^2 + p_2^2 + p_3^2) \\
&= p_1p_3 + q_1q_3 - p_1p_3 - q_1q_3 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
B_2 &= p_2p_3 - p_1p_2q_1q_3 - p_2^2q_2q_3 - p_2p_3q_3^2 + q_2q_3 - p_1p_3q_1q_2 \\
&\quad - p_2p_3q_2^2 - p_3^2q_2q_3 - p_1^2q_2q_3 + p_1p_3q_1q_2 + p_1p_2q_1q_3 - p_2p_3q_1^2 \\
&= p_2p_3 - p_2p_3(q_1^2 + q_2^2 + q_3^2) + q_2q_3 - q_2q_3(p_1^2 + p_2^2 + p_3^2)
\end{aligned}$$

$$= p_2p_3 + q_2q_3 - p_2p_3 - q_2q_3$$

$$= 0,$$

and

$$\begin{aligned}
C_2 =& -p_3^2 + p_1p_3q_1q_3 + p_2p_3q_2q_3 + p_3^2q_3^2 - q_3^2 + p_1p_3q_1q_3 \\
&+ p_2p_3q_2q_3 + p_3^2q_3^2 + p_1p_2q_1q_2 - p_1^2q_2^2 - p_2^2q_1^2 + p_1p_2q_1q_2 \\
=& \, 2(p_1p_2q_1q_2 + p_1p_3q_1q_3 + p_2p_3q_2q_3) - (1 - p_1^2 - p_2^2) - (1 - q_1^2 - q_2^2) \\
&+ 2(1 - p_1^2 - p_2^2)(1 - q_1^2 - q_2^2) - p_1^2q_2^2 - p_2^2q_1^2 \\
=& \, 2(p_1p_2q_1q_2 + p_1p_3q_1q_3 + p_2p_3q_2q_3) - 2 + p_1^2 + p_2^2 + q_1^2 + q_2^2 - p_1^2q_2^2 - p_2^2q_1^2 \\
&+ 2 - 2p_1^2 - 2p_2^2 - 2q_1^2 + 2p_1^2q_1^2 + 2p_2^2q_1^2 - 2q_2^2 + 2p_1^2q_2^2 + 2p_2^2q_2^2 \\
=& \, 2(p_1p_2q_1q_2 + p_1p_3q_1q_3 + p_2p_3q_2q_3) + 2(p_1^2q_1^2 + p_2^2q_2^2) \\
&- p_1^2 - p_2^2 - q_1^2 - q_2^2 + p_1^2q_2^2 + p_2^2q_1^2.
\end{aligned}$$

Moreover,

$$L_1' = A_3u_x + B_3u_y + C_3,$$

where

$$\begin{aligned}
A_3 =& \, p_1^2q_1q_2 + p_1p_2q_2^2 + p_1p_3q_2q_3 - p_1p_2 + p_1p_2q_1^2 + p_2^2q_1q_2 \\
&+ p_2p_3q_1q_3 - q_1q_2 + p_1p_2q_3^2 - p_1p_3q_2q_3 - p_2p_3q_1q_3 + p_3^2q_1q_2 \\
=& \, p_1p_2(q_1^2 + q_2^2 + q_3^2) - p_1p_2 - q_1q_2 + q_1q_2(p_1^2 + p_2^2 + p_3^2) \\
=& \, p_1p_2 + q_1q_2 - p_1p_2 - q_1q_2 \\
=& \, 0,
\end{aligned}$$

$$\begin{aligned}
B_3 =& \, p_1p_2q_1q_2 + p_2^2q_2^2 + p_2p_3q_2q_3 - p_2^2 + p_1p_2q_1q_2 + p_2^2q_2^2 \\
&+ p_2p_3q_2q_3 - q_2^2 - p_1^2q_3^2 + p_1p_3q_1q_3 + p_1p_3q_1q_3 - p_3^2q_1^2 \\
=& \, 2(p_1p_2q_1q_2 + p_2p_3q_2q_3 + p_1p_3q_1q_3) + 2p_2^2q_2^2 - p_2^2 - q_2^2 \\
&- p_1^2(1 - q_1^2 - q_2^2) - q_1^2(1 - p_1^2 - p_2^2) \\
=& \, 2(p_1p_2q_1q_2 + p_2p_3q_2q_3 + p_1p_3q_1q_3) + 2p_2^2q_2^2 - p_2^2 - q_2^2
\end{aligned}$$

$$
\begin{aligned}
&- p_1^2 + p_1^2 q_1^2 + p_1^2 q_2^2 - q_1^2 + p_1^2 q_1^2 + p_2^2 q_1^2 \\
&= 2(p_1p_2q_1q_2 + p_2p_3q_2q_3 + p_1p_3q_1q_3) + 2(p_1^2q_1^2 + p_2^2q_2^2) \\
&\quad - p_1^2 - p_2^2 - q_1^2 - q_2^2 + p_1^2q_2^2 + p_2^2q_1^2,
\end{aligned}
$$

and

$$
\begin{aligned}
C_3 =& - p_1p_3q_1q_2 - p_2p_3q_2^2 - p_3^2q_2q_3 + p_2p_3 - p_1p_2q_1q_3 - p_2^2q_2q_3 \\
&- p_2p_3q_3^2 + q_2q_3 + p_1p_2q_1q_3 - p_1^2q_2q_3 - p_2p_3q_1^2 + p_1p_3q_1q_2 \\
=& \; p_2p_3 - p_2p_3(q_1^2 + q_2^2 + q_3^2) + q_2q_3 - q_2q_3(p_1^2 + p_2^2 + p_3^2) \\
=& \; p_2p_3 + q_2q_3 - p_2p_3 - q_2q_3 \\
=& \; 0.
\end{aligned}
$$

Now, substituting the calculated values of $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$, $A_3$, $B_3$, and $C_3$ in the right-hand sides of $(A.2.1)$ and $(A.2.2)$, we get

$$
\frac{(q_1\langle p|q\rangle - p_1)E_1 + (p_1\langle p|q\rangle - q_1)E_2 + (p_3q_2 - p_2q_3)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\sqrt{\Lambda}} = u_x
$$

and

$$
\frac{(q_2\langle p|q\rangle - p_2)E_1 + (p_2\langle p|q\rangle - q_2)E_2 + (p_1q_3 - p_3q_1)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 + (p_1q_2 - p_2q_1)\sqrt{\Lambda}} = u_y.
$$

Assume now that $\varepsilon(x, y) = -1$. Then clearly $\operatorname{sgn}\varepsilon \operatorname{sgn}\theta < 0$. By Lemma 2.5 and (2.2)

$$
\frac{(q_1\langle p|q\rangle - p_1)E_1 + (p_1\langle p|q\rangle - q_1)E_2 - (p_3q_2 - p_2q_3)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 - (p_1q_2 - p_2q_1)\sqrt{\Lambda}} = \frac{M_1}{M_2} \qquad (A.2.3)
$$

and

$$\frac{(q_2\langle p|q\rangle - p_2)E_1 + (p_2\langle p|q\rangle - q_2)E_2 - (p_1q_3 - p_3q_1)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 - (p_1q_2 - p_2q_1)\sqrt{\Lambda}} = \frac{M_1'}{M_2}, \quad (A.2.4)$$

where

$$\begin{aligned}
M_1 = {} & p_1^2q_1^2u_x + p_1p_2q_1q_2u_x + p_1p_3q_1q_3u_x - p_1^2u_x \\
& + p_1p_2q_1^2u_y + p_2^2q_1q_2u_y + p_2p_3q_1q_3u_y - p_1p_2u_y \\
& - p_1p_3q_1^2 - p_2p_3q_1q_2 - p_3^2q_1q_3 + p_1p_3 \\
& + p_1^2q_1^2u_x + p_1p_2q_1q_2u_x + p_1p_3q_1q_3u_x - q_1^2u_x \\
& + p_1^2q_1q_2u_y + p_1p_2q_2^2u_y + p_1p_3q_2q_3u_y - q_1q_2u_y \\
& - p_1^2q_1q_3 - p_1p_2q_2q_3 - p_1p_3q_3^2 + q_1q_3 \\
& - p_2p_3q_2q_3u_x + p_3^2q_2^2u_x + p_1p_3q_2q_3u_y - p_3^2q_1q_2u_y - p_2p_3q_1q_2 + p_1p_3q_2^2 \\
& + p_2^2q_3^2u_x - p_2p_3q_2q_3u_x - p_1p_2q_3^2u_y + p_2p_3q_1q_3u_y + p_2^2q_1q_3 - p_1p_2q_2q_3,
\end{aligned}$$

$$\begin{aligned}
M_2 = {} & p_1p_3u_x - p_1^2q_1q_3u_x - p_1p_2q_2q_3u_x - p_1p_3q_3^2u_x \\
& + p_2p_3u_y - p_1p_2q_1q_3u_y - p_2^2q_2q_3u_y - p_2p_3q_3^2u_y \\
& - p_3^2 + p_1p_3q_1q_3 + p_2p_3q_2q_3 + p_3^2q_3^2 \\
& + q_1q_3u_x - p_1p_3q_1^2u_x - p_2p_3q_1q_2u_x - p_3^2q_1q_3u_x \\
& + q_2q_3u_y - p_1p_3q_1q_2u_y - p_2p_3q_2^2u_y - p_3^2q_2q_3u_y \\
& - q_3^2 + p_1p_3q_1q_3 + p_2p_3q_2q_3 + p_3^2q_3^2 \\
& - p_1p_2q_2q_3u_x + p_1p_3q_2^2u_x + p_1^2q_2q_3u_y - p_1p_3q_1q_2u_y - p_1p_2q_1q_2 + p_1^2q_2^2 \\
& + p_2^2q_1q_3u_x - p_2p_3q_1q_2u_x - p_1p_2q_1q_3u_y + p_2p_3q_1^2u_y + p_2^2q_1^2 - p_1p_2q_1q_2,
\end{aligned}$$

and

$$\begin{aligned}
M_1' = {} & p_1^2q_1q_2u_x + p_1p_2q_2^2u_x + p_1p_3q_2q_3u_x - p_1p_2u_x \\
& + p_1p_2q_1q_2u_y + p_2^2q_2^2u_y + p_2p_3q_2q_3u_y - p_2^2u_y \\
& - p_1p_3q_1q_2 - p_2p_3q_2^2 - p_3^2q_2q_3 + p_2p_3 \\
& + p_1p_2q_1^2u_x + p_2^2q_1q_2u_x + p_2p_3q_1q_3u_x - q_1q_2u_x
\end{aligned}$$

$$+ p_1p_2q_1q_2u_y + p_2^2q_2^2u_y + p_2p_3q_2q_3u_y - q_2^2u_y$$
$$- p_1p_2q_1q_3 - p_2^2q_2q_3 - p_2p_3q_3^2 + q_2q_3$$
$$- p_1p_2q_3^2u_x + p_1p_3q_2q_3u_x + p_1^2q_3^2u_y - p_1p_3q_1q_3u_y - p_1p_2q_1q_3 + p_1^2q_2q_3$$
$$+ p_2p_3q_1q_3u_x - p_3^2q_1q_2u_x - p_1p_3q_1q_3u_y + p_3^2q_1^2u_y + p_2p_3q_1^2 - p_1p_3q_1q_2.$$

It is clear that $M_1$ takes the form

$$M_1 = A_1'u_x + B_1'u_y + C_1',$$

where

$$\begin{aligned}
A_1' &= p_1^2q_1^2 + p_1p_2q_1q_2 + p_1p_3q_1q_3 - p_1^2 + p_1^2q_1^2 + p_1p_2q_1q_2\\
&\quad + p_1p_3q_1q_3 - q_1^2 - p_2p_3q_2q_3 + p_3^2q_2^2 + p_2^2q_3^2 - p_2p_3q_2q_3\\
&= p_1^2(1 - q_2^2 - q_3^2) - p_1^2 + 2p_1p_2q_1q_2 + 2p_1p_3q_1q_3 - 2p_2p_3q_2q_3\\
&\quad + q_1^2(1 - p_2^2 - p_3^2) - q_1^2 + p_3^2q_2^2 + p_2^2q_3^2\\
&= p_1^2 - p_1^2q_2^2 - p_1^2q_3^2 - p_1^2 + p_3^2q_2^2 + 2p_1p_2q_1q_2 + q_1^2\\
&\quad - p_2^2q_1^2 - p_3^2q_1^2 - q_1^2 + p_2^2q_3^2 + 2p_1p_3q_1q_3 - 2p_2p_3q_2q_3\\
&= - p_1^2q_2^2 - p_2^2q_1^2 + 2p_1p_2q_1q_2 + p_3^2q_2^2 + p_2^2q_3^2 - 2p_2p_3q_2q_3\\
&\quad - p_1^2q_3^2 - p_3^2q_1^2 + 2p_1p_3q_1q_3\\
&= (p_2q_3 - p_3q_2)^2 - (p_1q_2 - p_2q_1)^2 - (p_1q_3 - p_3q_1)^2\\
&= a^2 - b^2 - c^2,
\end{aligned}$$

$$\begin{aligned}
B_1' &= p_1p_2q_1^2 + p_2^2q_1q_2 + p_2p_3q_1q_3 - p_1p_2 + p_1^2q_1q_2 + p_1p_2q_2^2\\
&\quad + p_1p_3q_2q_3 - q_1q_2 + p_1p_3q_2q_3 - p_3^2q_1q_2 - p_1p_2q_3^2 + p_2p_3q_1q_3\\
&= q_1q_2(p_1^2 + p_2^2) - q_1q_2 - p_3^2q_1q_2 + p_3^2q_1q_2 - p_3^2q_1q_2 + 2p_1p_3q_2q_3\\
&\quad + p_1p_2(q_1^2 + q_2^2) - p_1p_2 - p_1p_2q_3^2 + p_1p_2q_3^3 - p_1p_2q_3^2 + 2p_2p_3q_1q_3\\
&= q_1q_2(p_1^2 + p_2^2 + p_3^2) - q_1q_2 - 2p_3^2q_1q_2 + 2p_1p_3q_2q_3\\
&\quad + p_1p_2(q_1^2 + q_2^2 + q_3^2) - p_1p_2 - 2p_1p_2q_3^2 + 2p_2p_3q_1q_3
\end{aligned}$$

$$
\begin{aligned}
&= 2p_1p_3q_2q_3 - 2p_3^2q_1q_2 + 2p_2p_3q_1q_3 - 2p_1p_2q_3^2 \\
&= 2\big[(p_3q_2(p_1q_3 - p_3q_1) + p_2q_3(p_3q_1 - p_1q_3)\big] \\
&= 2(p_1q_3 - p_3q_1)(p_3q_2 - p_2q_3) \\
&= 2ac,
\end{aligned}
$$

and

$$
\begin{aligned}
C_1' =& - p_1p_3q_1^2 - p_2p_3q_1q_2 - p_3^2q_1q_3 + p_1p_3 - p_1^2q_1q_3 - p_1p_2q_2q_3 \\
&\quad - p_1p_3q_3^2 + q_1q_3 - p_2p_3q_1q_2 + p_1p_3q_2^2 + p_2^2q_1q_3 - p_1p_2q_2q_3 \\
=& - p_1p_3(q_1^2 + q_3^2) + p_1p_3 + p_1p_3q_2^2 - p_1p_3q_2^2 + p_1p_3q_2^2 - 2p_1p_2q_2q_3 \\
&\quad - q_1q_3(p_1^2 + p_3^2) + q_1q_3 + p_2^2q_1q_3 - p_2^2q_1q_3 + p_2^2q_1q_3 - 2p_2p_3q_1q_2 \\
=& - p_1p_3(q_1^2 + q_2^2 + q_3^2) + p_1p_3 + 2p_1p_3q_2^2 - 2p_1p_2q_2q_3 \\
&\quad - q_1q_3(p_1^2 + p_2^2 + p_3^2) + q_1q_3 + 2p_2^2q_1q_3 - 2p_2p_3q_1q_2 \\
=& 2(p_1p_3q_2^2 - p_1p_2q_2q_3 + p_2^2q_1q_3 - p_2p_3q_1q_2) \\
=& 2\big[p_1q_2(p_3q_2 - p_2q_3) + p_2q_1(p_2q_3 - p_3q_2)\big] \\
=& 2(p_3q_2 - p_2q_3)(p_1q_2 - p_2q_1) \\
=& 2ab.
\end{aligned}
$$

Similarly, $M_2$ takes the form

$$
M_2 = A_2'u_x + B_2'u_y + C_2',
$$

where

$$
\begin{aligned}
A_2' =& p_1p_3 - p_1^2q_1q_3 - p_1p_2q_2q_3 - p_1p_3q_3^2 + q_1q_3 - p_1p_3q_1^2 \\
&\quad - p_2p_3q_1q_2 - p_3^2q_1q_3 - p_1p_2q_2q_3 + p_1p_3q_2^2 + p_2^2q_1q_3 - p_2p_3q_1q_2 \\
=& p_1p_3 - p_1p_3(q_1^2 + q_3^2) + p_1p_3q_2^2 - p_1p_3q_2^2 + p_1p_3q_2^2 - 2p_1p_2q_1q_3 \\
&\quad + q_1q_3 - q_1q_3(p_1^2 + p_3^2) + p_2^2q_1q_3 - p_2^2q_1q_3 + p_2^2q_1q_3 - 2p_2p_3q_1q_2 \\
=& p_1p_3 - p_1p_3(q_1^2 + q_2^2 + q_3^2) + 2p_1p_3q_2^2 - 2p_1p_2q_2q_3 \\
&\quad + q_1q_3 - q_1q_3(p_1^2 + p_2^2 + p_3^2) + 2p_2^2q_1q_3 - 2p_1p_3q_1q_2
\end{aligned}
$$

$$
\begin{aligned}
&= 2(p_1p_3q_2^2 - p_1p_2q_2q_3 + p_2^2q_1q_3 - p_2p_3q_1q_2)\\
&= 2\big[p_1q_2(p_3q_2 - p_2q_3) + p_2q_1(p_2q_3 - p_3q_2)\big]\\
&= 2(p_3q_2 - p_2q_3)(p_1q_2 - p_2q_1)\\
&= 2ab,
\end{aligned}
$$

$$
\begin{aligned}
B_2' &= p_2p_3 - p_1p_2q_1q_3 - p_2^2q_2q_3 - p_2p_3q_3^2 + q_2q_3 - p_1p_3q_1q_2\\
&\quad - p_2p_3q_2^2 - p_3^2q_2q_3 + p_1^2q_2q_3 - p_1p_3q_1q_2 - p_1p_2q_1q_3 + p_2p_3q_1^2\\
&= p_2p_3 - p_2p_3(q_2^2 + q_3^2) + p_2p_3q_1^2 - p_2p_3q_1^2 + p_2p_3q_1^2 - 2p_1p_2q_1q_3\\
&\quad + q_2q_3 - q_2q_3(p_2^2 + p_3^2) + p_1^2q_2q_3 - p_1^2q_2q_3 + p_1^2q_2q_3 - 2p_1p_3q_1q_2\\
&= p_2p_3 - p_2p_3(q_1^2 + q_2^2 + q_3^2) + 2p_2p_3q_1^2 - 2p_1p_2q_1q_3\\
&\quad + q_2q_3 - q_2q_3(p_1^2 + p_2^2 + p_3^2) + 2p_1^2q_2q_3 - 2p_1p_3q_1q_2\\
&= 2(p_2p_3q_1^2 - p_1p_2q_1q_3 + p_1^2q_2q_3 - p_1p_3q_1q_2)\\
&= 2\big[p_2q_1(p_3q_1 - p_1q_3) + p_1q_2(p_1q_3 - p_3q_1)\big]\\
&= 2(p_3q_1 - p_1q_3)(p_2q_1 - p_1q_2)\\
&= 2bc,
\end{aligned}
$$

and

$$
\begin{aligned}
C_2' &= -p_3^2 + p_1p_3q_1q_3 + p_2p_3q_2q_3 + p_3^2q_3^2 - q_3^2 + p_1p_3q_1q_3\\
&\quad + p_2p_3q_2q_3 + p_3^2q_3^2 - p_1p_2q_1q_2 + p_1^2q_2^2 + p_2^2q_1^2 - p_1p_2q_1q_2\\
&= 2p_1p_3q_1q_3 - p_3^2 + 2p_2p_3q_2q_3 + 2p_3^2q_3^2 - q_3^2 - 2p_1p_2q_1q_2 + p_1^2q_2^2 + p_2^2q_1^2\\
&= p_3^2(1 - q_1^2 - q_2^2) - p_3^2 + q_3^2(1 - p_1^2 - p_2^2) - q_3^2 + p_1^2q_2^2 + p_2^2q_1^2\\
&\quad + 2p_1p_3q_1q_3 + 2p_2p_3q_2q_3 - 2p_1p_2q_1q_2\\
&= p_1^2q_2^2 + p_2^2q_1^2 - 2p_1p_2q_1q_2 - (p_3^2q_1^2 + p_1^2q_3^2 - 2p_1p_3q_1q_3)\\
&\quad - (p_3^2q_2^2 + p_2^2q_3^2 - 2p_2p_3q_2q_3)\\
&= (p_1q_2 - p_2q_1)^2 - (p_3q_1 - p_1q_3)^2 - (p_3q_2 - p_2q_3)^2\\
&= b^2 - a^2 - c^2.
\end{aligned}
$$

Moreover,

$$M_1' = A_3' u_x + B_3' u_y + C_3',$$

where

$$\begin{aligned}
A_3' &= p_1^2 q_1 q_2 + p_1 p_2 q_2^2 + p_1 p_3 q_2 q_3 - p_1 p_2 + p_1 p_2 q_1^2 + p_2^2 q_1 q_2 \\
&\quad + p_2 p_3 q_1 q_3 - q_1 q_2 - p_1 p_2 q_3^2 + p_1 p_3 q_2 q_3 + p_2 p_3 q_1 q_3 - p_3^2 q_1 q_2 \\
&= - p_1 p_2 + p_1 p_2 (q_1^2 + q_2^2) - p_1 p_2 q_3^2 + p_1 p_2 q_3^2 - p_1 p_2 q_3^2 + 2 p_1 p_3 q_2 q_3 \\
&\quad - q_1 q_2 + q_1 q_2 (p_1^2 + p_2^2) - p_3^2 q_1 q_2 + p_3^2 q_1 q_2 - p_3^2 q_1 q_2 + 2 p_2 p_3 q_1 q_3 \\
&= - p_1 p_2 + p_1 p_2 (q_1^2 + q_2^2 + q_3^3) - 2 p_1 p_2 q_3^2 + 2 p_1 p_3 q_2 q_3 \\
&\quad - q_1 q_2 + q_1 q_2 (p_1^2 + p_2^2 + p_3^2) - 2 p_3^2 q_1 q_2 + 2 p_2 p_3 q_1 q_3 \\
&= 2 (p_1 p_3 q_2 q_3 - p_1 p_2 q_3^2 + p_2 p_3 q_1 q_3 - p_3^2 q_1 q_2) \\
&= 2 \big[ p_1 q_3 (p_3 q_2 - p_2 q_3) + p_3 q_1 (p_2 q_3 - p_3 q_2) \big] \\
&= 2 (p_3 q_2 - p_2 q_3)(p_1 q_3 - p_3 q_1) \\
&= 2ac,
\end{aligned}$$

$$\begin{aligned}
B_3' &= p_1 p_2 q_1 q_2 + p_2^2 q_2^2 + p_2 p_3 q_2 q_3 - p_2^2 + p_1 p_2 q_1 q_2 + p_2^2 q_2^2 \\
&\quad + p_2 p_3 q_2 q_3 - q_2^2 + p_1^2 q_3^2 - p_1 p_3 q_1 q_3 - p_1 p_3 q_1 q_3 + p_3^2 q_1^2 \\
&= 2 p_1 p_2 q_1 q_2 + q_2^2 (1 - p_1^2 - p_3^2) + 2 p_2 p_3 q_2 q_3 - p_2^2 - q_2^2 + p_1^2 q_3^2 \\
&\quad + p_3^2 q_1^2 - 2 p_1 p_3 q_1 q_3 + p_2^2 (1 - q_1^2 - q_3^2) \\
&= 2 p_1 p_2 q_1 q_2 + 2 p_2 p_3 q_2 q_3 - 2 p_1 p_3 q_1 q_3 + p_1^2 q_3^2 + p_3^2 q_1^2 - q_2^2 - p_2^2 + q_2^2 \\
&\quad - p_1^2 q_2^2 - p_3^2 q_2^2 + p_2^2 - p_2^2 q_1^2 - p_2^2 q_3^2 \\
&= p_3^2 q_1^2 + p_1^2 q_3^2 - 2 p_1 p_3 q_1 q_3 - (p_1^2 q_2^2 + p_2^2 q_1^2 - 2 p_1 p_2 q_1 q_2) \\
&\quad - (p_3^2 q_2^2 + p_2^2 q_3^2 - 2 p_2 p_3 q_2 q_3) \\
&= (p_1 q_3 - p_3 q_1)^2 - (p_1 q_2 - p_2 q_1)^2 - (p_3 q_2 - p_2 q_3)^2 \\
&= c^2 - a^2 - b^2,
\end{aligned}$$

and

$$
\begin{aligned}
C_3' = &- p_1p_3q_1q_2 - p_2p_3q_2^2 - p_3^2q_2q_3 + p_2p_3 - p_1p_2q_1q_3 - p_2^2q_2q_3 \\
&- p_2p_3q_3^2 + q_2q_3 - p_1p_2q_1q_3 + p_1^2q_2q_3 + p_2p_3q_1^2 - p_1p_3q_1q_2 \\
= &\ p_2p_3 - p_2p_3(q_2^2 + q_3^2) + p_2p_3q_1^2 - p_2p_3q_1^2 + p_2p_3q_1^2 - 2p_1p_3q_1q_2 \\
&+ q_2q_3 - q_2q_3(p_2^2 + q_3^2) + p_1^2q_2q_3 - p_1^2q_2q_3 + p_1^2q_2q_3 - 2p_1p_2q_1q_3 \\
= &\ p_2p_3 - p_2p_3(q_1^2 + q_2^2 + q_3^2) + 2p_2p_3q_1^2 - 2p_1p_3q_1q_2 \\
&+ q_2q_3 - q_2q_3(p_1^2 + p_2^2 + p_3^2) + 2p_1^2q_2q_3 - 2p_1p_2q_1q_3 \\
= &\ 2(p_2p_3q_1^2 - p_1p_3q_1q_2 + p_1^2q_2q_3 - p_1p_2q_1q_3) \\
= &\ 2\big[p_3q_1(p_2q_1 - p_1q_2) + p_1q_3(p_1q_2 - p_2q_1)\big] \\
= &\ 2(p_2q_1 - p_1q_2)(p_3q_1 - p_1q_3) \\
= &\ 2bc.
\end{aligned}
$$

Now, substituting the calculated values of $A_1'$, $B_1'$, $C_1'$, $A_2'$, $B_2'$, $C_2'$, $A_3'$, $B_3'$, and $C_3'$ in the right-hand sides of $(A.2.3)$ and $(A.2.4)$, we get

$$
\frac{(q_1\langle p|q\rangle - p_1)E_1 + (p_1\langle p|q\rangle - q_1)E_2 - (p_3q_2 - p_2q_3)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 - (p_1q_2 - p_2q_1)\sqrt{\Lambda}}
= \frac{(a^2 - b^2 - c^2)u_x + 2acu_y + 2ab}{2abu_x + 2bcu_y + b^2 - a^2 - c^2}
$$

and

$$
\frac{(q_2\langle p|q\rangle - p_2)E_1 + (p_2\langle p|q\rangle - q_2)E_2 - (p_1q_3 - p_3q_1)\sqrt{\Lambda}}{(p_3 - q_3\langle p|q\rangle)E_1 + (q_3 - p_3\langle p|q\rangle)E_2 - (p_1q_2 - p_2q_1)\sqrt{\Lambda}}
= \frac{2acu_x + (c^2 - a^2 - b^2)u_y + 2bc}{2abu_x + 2bcu_y + b^2 - a^2 - c^2}.
$$

A similar argument applies to the case in which

$$
\theta = u_x(p_2q_3 - p_3q_2) - u_y(p_1q_3 - p_3q_1) + p_2q_1 - p_1q_2 < 0. \quad \square
$$

# Appendix 3

In this appendix we find a formula for the function $\phi(t)$ that appeared in the course of the proof of Theorem 2.19.

**Lemma 2.20.** *Let*

$$v_1 = p_2q_3 - p_3q_2, \quad v_2 = p_3q_1 - p_1q_3, \quad \text{and} \quad v_3 = p_1q_2 - p_2q_1$$

*be such that $v_1^2 + v_2^2 > 0$. Suppose that $E_1$ and $E_2$ are continuous functions over an open convex $\Omega$ and that*

$$v_2\phi'(v_2x - v_1y) = \frac{-v_1v_3}{v_1^2+v_2^2} + \frac{\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)E_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2}, \tag{A.3.1}$$

$$v_1\phi'(v_2x - v_1y) = \frac{v_2v_3}{v_1^2+v_2^2} - \frac{\|p\|(q_2\langle p|q\rangle - p_2\|q\|^2)E_1 + \|q\|(p_2\langle p|q\rangle - q_2\|p\|^2)E_2}{\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)E_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)E_2}.$$

*Let*

$$\widetilde{E}_1(s) = E_1\Big(\frac{v_2s}{v_1^2+v_2^2}, \frac{-v_1s}{v_1^2+v_2^2}\Big) \quad \text{and} \quad \widetilde{E}_2(s) = E_2\Big(\frac{v_2s}{v_1^2+v_2^2}, \frac{-v_1s}{v_1^2+v_2^2}\Big).$$

*Then $\phi$, up to a constant, is given by*

$$\phi(t) = \int_{t_0}^{t} \frac{v_2\big[\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)\widetilde{E}_1(s) + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)\widetilde{E}_2(s)\big]}{(v_1^2+v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(s)\big]}\,ds$$

$$+ \int_{t_0}^{t} \frac{v_1\big[\|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)\widetilde{E}_2(s)\big]}{(v_1^2+v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(s)\big]}\,ds.$$

**Proof.** Let

$$A = \|p\|\widetilde{E}_1,\ B = \|q\|\widetilde{E}_2,\ a = \langle p|q\rangle,\ b = \|p\|^2,\ \text{and}\ c = \|q\|^2. \tag{A.3.2}$$

Suppose that $v_2 \neq 0$. Then the first equation in $(A.3.1)$ implies that

$$\phi'(t) = -\frac{v_1 v_3}{v_2(v_1^2+v_2^2)} + \frac{A(aq_1 - cp_1) + B(ap_1 - bq_1)}{v_2\big[A(cp_3 - aq_3) + B(bq_3 - ap_3)\big]}.$$

This can be rewritten in the form

$$\phi'(t) = \frac{I_1 + I_2 + v_2^2\big[A(aq_1 - cp_1) + B(ap_1 - bq_1)\big]}{v_2(v_1^2+v_2^2)\big[A(cp_3 - aq_3) + B(bq_3 - ap_3)\big]},$$

where $I_1$ and $I_2$ are defined as

$$I_1 = A\big[-v_1v_3(cp_3 - aq_3) + v_1^2(aq_1 - cp_1)\big],$$

$$I_2 = B\big[-v_1v_3(bq_3 - ap_3) + v_1^2(ap_1 - bq_1)\big].$$

Since

$$v_1^2 = p_2^2q_3^2 + p_3^2q_2^2 - 2p_2p_3q_2q_3,$$

$$v_1v_3 = p_1p_2q_2q_3 - p_1p_3q_2^2 - p_2^2q_1q_3 + p_2p_3q_1q_2,$$

it follows that

$$\begin{aligned}
I_1 &= A(-cp_1p_2p_3q_2q_3 + cp_1p_3^2q_2^2 + cp_2^2p_3q_1q_3 - cp_2p_3^2q_1q_2 \\
&\quad + ap_1p_2q_2q_3^2 - ap_1p_3q_2^2q_3 - ap_2^2q_1q_3^2 + ap_2p_3q_1q_2q_3 \\
&\quad + ap_2^2q_1q_3^2 + ap_3^2q_1q_2^2 - 2ap_2p_3q_1q_2q_3 \\
&\quad - cp_1p_2^2q_3^2 - cp_1p_3^2q_2^2 + 2cp_1p_2p_3q_2q_3) \\
&= A(cp_2^2p_3q_1q_3 - cp_2p_3^2q_1q_2 + ap_1p_2q_2q_3^2 - ap_1p_3q_2^2q_3 \\
&\quad + ap_3^2q_1q_2^2 - ap_2p_3q_1q_2q_3 - cp_1p_2^2q_3^2 + cp_1p_2p_3q_2q_3) \\
&= A\big[(cp_2^2p_3q_1q_3 - cp_1p_2^2q_3^2) - (cp_2p_3^2q_1q_2 - cp_1p_2p_3q_2q_3) \\
&\quad - (ap_2p_3q_1q_2q_3 - ap_1p_2q_2q_3^2) + (ap_3^2q_1q_2^2 - ap_1p_3q_2^2q_3)\big] \\
&= A\big[c(p_3q_1 - p_1q_3)p_2^2q_3 - c(p_3q_1 - p_1q_3)p_2p_3q_2 - a(p_3q_1 - p_1q_3)p_2q_2q_3 \\
&\quad + a(p_3q_1 - p_1q_3)p_3q_2^2\big] \\
&= A(p_3q_1 - p_1q_3)\big[cp_2(p_2q_3 - p_3q_2) + aq_2(p_3q_2 - p_2q_3)\big] \\
&= A(p_3q_1 - p_1q_3)(p_2q_3 - p_3q_2)(cp_2 - aq_2) \\
&= Av_1v_2(cp_2 - aq_2).
\end{aligned}$$

Analogously

$$\begin{aligned}
I_2 = \; & B(-bp_1p_2q_2q_3^2 + bp_1p_3q_2^2q_3 + bp_2^2q_1q_3^2 - bp_2p_3q_1q_2q_3 \\
& + ap_1p_2p_3q_2q_3 - ap_1p_3^2q_2^2 - ap_2^2p_3q_1q_3 + ap_2p_3^2q_1q_2 \\
& + ap_1p_2^2q_3^2 + ap_1p_3^2q_2^2 - 2ap_1p_2p_3q_2q_3 \\
& - bp_2^2q_1q_3^2 - bp_3^2q_1q_2^2 + 2bp_2p_3q_1q_2q_3) \\
= \; & B(-bp_1p_2q_2q_3^2 + bp_1p_3q_2^2q_3 - ap_2^2p_3q_1q_3 + ap_2p_3^2q_1q_2 \\
& + ap_1p_2^2q_3^2 - ap_1p_2p_3q_2q_3 - bp_3^2q_1q_2^2 + bp_2p_3q_1q_2q_3) \\
= \; & B\big[-(ap_2^2p_3q_1q_3 - ap_1p_2^2q_3^2) - (bp_3^2q_1q_2^2 - bp_1p_3q_2^2q_3) \\
& + (ap_2p_3^2q_1q_2 - ap_1p_2p_3q_2q_3) + (bp_2p_3q_1q_2q_3 - bp_1p_2q_2q_3^2)\big] \\
= \; & B\big[-a(p_3q_1 - p_1q_3)p_2^2q_3 - b(p_3q_1 - p_1q_3)p_3q_2^2 + a(p_3q_1 - p_1q_3)p_2p_3q_2 \\
& + b(p_3q_1 - p_1q_3)p_2q_2q_3\big] \\
= \; & B(p_3q_1 - p_1q_3)\big[bq_2(p_2q_3 - p_3q_2) + ap_2(p_3q_2 - p_2q_3)\big] \\
= \; & B(p_3q_1 - p_1q_3)(p_2q_3 - p_3q_2)(bq_2 - ap_2) \\
= \; & Bv_1v_2(bq_2 - ap_2).
\end{aligned}$$

Now taking account of $(A.3.2)$, we see that

$$\begin{aligned}
\phi'(t) &= \frac{Av_1(cp_2 - aq_2) + Bv_1(bq_2 - ap_2) + v_2\big[A(aq_1 - cp_1) + B(ap_1 - bq_1)\big]}{(v_1^2 + v_2^2)\big[A(cp_3 - aq_3) + B(bq_3 - ap_3)\big]} \\
&= \frac{v_2\big[A(aq_1 - cp_1) + B(ap_1 - bq_1)\big] + v_1\big[A(cp_2 - aq_2) + B(bq_2 - ap_2)\big]}{(v_1^2 + v_2^2)\big[A(cp_3 - aq_3) + B(bq_3 - ap_3)\big]} \\
&= \frac{v_2\big[\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)\widetilde{E}_1 + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)\widetilde{E}_2\big]}{(v_1^2 + v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2\big]} \\
&\quad + \frac{v_1\big[\|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)\widetilde{E}_1 + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)\widetilde{E}_2\big]}{(v_1^2 + v_2^2)\big[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1 + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2\big]}.
\end{aligned}$$

Thus function $\phi$, up to a constant, is given by

$$\phi(t) = \int_{t_0}^{t} \frac{v_2\left[\|p\|(q_1\langle p|q\rangle - p_1\|q\|^2)\widetilde{E}_1(s) + \|q\|(p_1\langle p|q\rangle - q_1\|p\|^2)\widetilde{E}_2(s)\right]}{(v_1^2+v_2^2)\left[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(s)\right]}\, ds$$

$$+ \int_{t_0}^{t} \frac{v_1\left[\|p\|(p_2\|q\|^2 - q_2\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_2\|p\|^2 - p_2\langle p|q\rangle)\widetilde{E}_2(s)\right]}{(v_1^2+v_2^2)\left[\|p\|(p_3\|q\|^2 - q_3\langle p|q\rangle)\widetilde{E}_1(s) + \|q\|(q_3\|p\|^2 - p_3\langle p|q\rangle)\widetilde{E}_2(s)\right]}\, ds.$$

The case $v_1 \neq 0$ is treated similarly by taking as a starting point the second equation in $(A.3.1)$. $\square$

# Supplement

In this supplement, we present an overview of the results from the joint papers [3], [4], [5], and [6] written by Brooks, Chojnacki and Kozera. The author of this thesis contributed approximately one third to this work, which was undertaken during his Ph. D. candidature. These results are a contribution to a single-image shape from shading and as such are of no direct relevance to the analysis of photometric stereo. As the supplement is of informal character, various theorems to be presented will not be accompanied by proofs.

## S.1. Introduction

Given an image, the natural question arises as to whether it actually corresponds to a physically-realisable shape. For Lambertian shading with an overhead point-source illumination, this reduces to the problem of solving

$$u_x^2 + u_y^2 = \mathcal{E}(x, y) \tag{S.1}$$

over a given domain. It was Horn who first posed this problem and who coined the term *impossible shading* for a brightness pattern that could not be the image of a smooth surface. Below, we present two different classes of images for which there are no genuine shapes. Initially, we reveal a class of images for which only unbounded (and therefore physically-unrealisable) shapes exist. Next, we present a class of images exhibiting shading for which neither bounded nor unbounded shapes exist.

Given an image of some particular shape, another question arises as to whether it could be or could not be also an image of the other shape. Uniqueness of this kind has been demonstrated for equation (S.1) in the case where

$$\mathcal{E}(x, y) = \frac{x^2 + y^2}{1 - x^2 - y^2} \ .$$

Deift and Sylvester [8], and independently Brooks [2], proved that $\pm(1-x^2-y^2)^{1/2}+c$ are the only $C^2$ solutions to this equation over the unit disc $\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}$. All of these solutions are hemispherical in shape. Interestingly, this result fails in the class of $C^1$ solutions.

In an effort to obtain a more general result, Bruss [7], in perhaps the major work in the uniqueness area, asserted the following: if $R$ is a positive number, $D(R)$ is the disc in the $xy$-plane with radius $R$ centred at the origin, and $f$ is a continuous function on $[0,R)$ of class $C^2$ over $(0,R)$ such that

(i) $f(0)=0$ and $f(r)>0$ for $0<r<R$,

(ii) $\lim_{r\to 0} f'(r)=0$, $\lim_{r\to 0} f''(r)$ exists and is positive,

(iii) $\lim_{r\to R} f(r)=+\infty$,

then all solutions of class $C^2$ to (1) in $D(R)$ with

$$\mathcal{E}(x,y)=f(\sqrt{x^2+y^2}) \tag{S.2}$$

take the form

$$\pm\int_0^{\sqrt{x^2+y^2}} \sqrt{f(\sigma)}\,d\sigma+k,$$

and so are circularly-symmetric with common shape. Here, conditions (i) and (ii) ensure that the origin is the only (singular) point at which $\mathcal{E}$ vanishes to second order, while condition (iii) implies that the Euclidean norm of the gradient of any solution to (S.1) diverges to infinity as the circumference of $D(R)$ is approached. As we shall show, this assertion is invalid. Specifically, we shall reveal a class of functions $f$, having the above properties, for which the corresponding eikonal equations have a bounded, non-circularly-symmetric solution of class $C^2$.

## S.2. Existence Results

### *S.2.1. Images Without Bounded Solution*

Let $R$ be either a positive number or $+\infty$. Let $f$ be a non-negative continuous function on the interval $[0,R)$ vanishing exactly at zero. Consider equation (S.1) with $\mathcal{E}$ given by (S.2). With this special form of $\mathcal{E}$, the class of circularly-symmetric solutions is readily determined. Each solution in this class takes the form $\pm U +$ const, where

$$U(x,y) = \int_0^{\sqrt{x^2+y^2}} \sqrt{f(\sigma)}\, d\sigma.$$

Note that it is critical that $f$ vanish at zero so as to ensure the differentiability of $U$ at the origin of the $xy$-plane. Our eikonal equation may also admit non-circularly-symmetric solutions. The function $u(x,y) = xy$ provides an example of such a solution when $f(r) = r^2$ and $R = +\infty$. Unlike the class of circularly-symmetric solutions, the class of all non-circularly-symmetric solutions is not easily specified.

A condition on $f$ guaranteeing that all solutions to the corresponding eikonal equation are unbounded may readily be formulated. Clearly, in the class of circularly-symmetric solutions, this sufficient condition is

$$\int_0^R \sqrt{f(\sigma)}\, d\sigma = +\infty. \tag{S.3}$$

It is less evident, though true, that the same condition is sufficient in the general case. In fact, we have the following:

**Theorem S.1.** *Let $f$ be a non-negative continuous function on $[0,R)$ vanishing exactly at zero and satisfying* (S.3). *Then there is no bounded $C^1$ solution in $D(R)$ to* (S.1) *with $\mathcal{E}$ given by* (S.2).

Interestingly, condition (S.3) is not only sufficient but also necessary for the unboundedness of all solutions to the equation in question. We have the following theorem:

**Theorem S.2.** *Let $f$ be a non-negative continuous function in $[0,R)$ vanishing exactly at zero and satisfying*

$$\int_0^R \sqrt{f(\sigma)}\, d\sigma < +\infty.$$

*Then every solution in $D(R)$ to* (S.1) *with $\mathcal{E}$ given by* (S.2) *is bounded. Moreover, if $u$ is any such solution, then*

$$\sup_{(x,y)\in D(R)} u(x,y) - \inf_{(x,y)\in D(R)} u(x,y) \le 2\int_0^R \sqrt{f(\sigma)}\, d\sigma.$$

Observe that whether the integral $\int_0^R \sqrt{f(\sigma)}\, d\sigma$ is finite or infinite depends exclusively on the behaviour of $f$ near $R$. The integral will be infinite if, for example, $f(r)$ diverges to infinity sufficiently rapidly as $r$ tends to $R$. This means that, in the context of real images of Lambertian surfaces illuminated by an overhead point-source, a circularly-symmetric image cannot be derived from a genuine shape if it gets dark too quickly as the image boundary is approached. Note also that the above integral may be finite or infinite under the condition that $R$ is finite and $\lim_{r \to R} f(r) = +\infty$, which implies that the Euclidean norm of the gradient of any solution to (S.1) diverges to infinity as the circumference of $D(R)$ is approached. This is of interest in that it relates to the familiar notion of the *occluding boundary.* The following examples show that the integral may be finite or infinite with the above condition being met: if $R = \pi/2$ and $f(r) = \tan^2 r$, then the integral is infinite, and so no bounded solutions to (S.1) can exist; on the other hand, if $R = 1$ and $f(r) = r^2(1 - r^2)^{-1}$ (the image of the unit hemisphere centered at the origin), then the integral is finite, and so all solutions to (S.1) must be bounded.

Comparison of Theorems S.1 and S.2 reveals the following remarkable dichotomy: either all solutions to equation (S.1) with $\mathcal{E}$ given by (S.2) are bounded, or all solutions are unbounded, according to whether the integral $\int_0^R \sqrt{f(\sigma)}\, d\sigma$ is finite or infinite, respectively. The question then arises as to whether there is an eikonal equation having both an unbounded and a bounded solution. This is answered in the affirmative when we note that, in the semidisc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x > 0\}$, the bounded function $\arctan(yx^{-1})$ and the unbounded function $\ln \sqrt{x^2 + y^2} + 1$ both satisfy the eikonal equation $u_x^2 + u_y^2 = (x^2 + y^2)^{-1}$.

### *S.2.2. Images Without Solution*

We now establish the existence of images $\mathcal{E}$ for which there is no solution to equation (S.1). In addition, we offer some insight into the result. The theorem presented below is a refinement of that due to Horn, Szeliski, and Yuille [14]; its proof, to be found in [3], elaborates an outline also due to Horn *et al.*

**Theorem S.3.** *Let $\Omega$ be a bounded open connected subset of the xy-plane with boundary $\partial\Omega$ being a piecewise $C^1$ curve of length $\ell_{\partial\Omega}$. Let $(x_0, y_0)$ be a point in $\Omega$ and $r$ be a positive number such that the closed disc $\bar{D}(x_0, y_0, r)$ of radius*

*$r$ centered at $(x_0, y_0)$ is contained in $\Omega$. Suppose $\mathcal{E}$ is a non-negative continuous function on the closure of $\Omega$, positive in $\Omega$, such that*

$$4r\sqrt{\mathcal{E}_1} > \ell_{\partial\Omega}\sqrt{\mathcal{E}_2}, \tag{S.4}$$

*where $\mathcal{E}_1 = \min\{\mathcal{E}(x,y) : (x,y) \in \bar{D}(x_0, y_0, r)\}$ and $\mathcal{E}_2 = \max\{\mathcal{E}(x,y) : (x,y) \in \partial\Omega\}$. Then there is no $C^1$ solution to* (S.1) *in $\Omega$.*

Note that the theorem is of local character: if $\Omega$ is a subset of a domain $\Delta$ and $\mathcal{E}$ is a non-negative function on $\Delta$ whose restriction to $\Omega$ satisfies (S.4) for some choice of $\bar{D}(x_0, y_0, r)$ in $\Omega$, then, obviously, there is no $C^1$ solution to (S.1) in $\Delta$. Reformulated in terms of Lambertian shading, this locality property can be expressed as saying that no genuine image can admit too dark a spot on too bright a background, assuming that the background does not contain a point having unit brightness. The precise balance between the qualifications "too dark" and "too bright" is, of course, given by condition (S.4).

## S.3. Uniqueness Results

### *S.3.1. Solutions Over Quadrants And Discs*

The construction of non-circularly-symmetric solutions to eikonal equations with $\mathcal{E}$ given by (S.2) will be divided into several steps. The graph of any such solution will take the form of a saddle having four regions of monotonicity spread out over four quadrants in the $xy$-plane determined by the lines $x = \pm y$. First, we shall construct a portion of a typical solution over the quadrant containing the positive $x$-halfaxis; the three remaining portions will easily be generated from this one. Next, we shall specify a class of functions $f$ for which the portions over all four quadrants can be smoothly pasted together and shall describe the corresponding process of gluing. Finally, we shall discuss the differentiability properties of the solutions obtained.

We now undertake the first stage of the construction.

**Theorem S.4.** *Let $R$ be either a positive number or $+\infty$. Let $f$ be a positive function of class $C^2$ on $(0, R)$ such that*

$$\lim_{r \to 0} f(r) = 0, \tag{S.5}$$

$$\lim_{r\to 0} \frac{f'(r)}{r} = 2, \tag{S.6}$$

*and*

$$r\big[f''(r)f(r) - \big(f'(r)\big)^2\big] + f(r)f'(r) \geq 0 \tag{S.7}$$

*for* $0 < r < R$. *Then there is a unique solution* $u$ *of class* $C^2$ *to* (S.1), *with* $\mathcal{E}$ *given by* (S.2), *defined over the quadrant*

$$Q_1(R) = \{(x,y) \in \mathbb{R}^2 : |y| < x,\ 0 < x < R\},$$

*such that* $u$ *is positive in the upper* $xy$*-halfplane and vanishes at the positive* $x$*-halfaxis. Moreover,* $u(x,-y) = -u(x,y)$ *for each* $(x,y)$ *in* $Q_1(R)$.

Proceeding to the next stage of the construction, let $R$ be either a positive number or $+\infty$, and let

$$\begin{aligned}
Q_2(R) &= \{(x,y) \in \mathbb{R}^2 : |x| < y, 0 < y < R\},\\
Q_3(R) &= \{(x,y) \in \mathbb{R}^2 : |y| < -x, -R < x < 0\},\\
Q_4(R) &= \{(x,y) \in \mathbb{R}^2 : |x| < -y, -R < y < 0\}.
\end{aligned}$$

Given a positive function $f$ of class $C^2$ on $(0,R)$ satisfying (S.5), (S.6), and (S.7), let $u$ be the solution to (S.1), with $\mathcal{E}$ given by (S.2), defined over $Q_1(R)$ that has the properties stated in Theorem S.4. Let

$$U(x,y) = \begin{cases}
u(x,y), & \text{if } (x,y) \in Q_1(R);\\
u(y,x), & \text{if } (x,y) \in Q_2(R);\\
u(-x,-y), & \text{if } (x,y) \in Q_3(R);\\
u(-y,-x), & \text{if } (x,y) \in Q_4(R);\\
\int_0^{\sqrt{2}|x|} \sqrt{f(\sigma)}\, d\sigma, & \text{if } -R < x = y < R;\\
-\int_0^{\sqrt{2}|x|} \sqrt{f(\sigma)}\, d\sigma, & \text{if } -R < x = -y < R.
\end{cases}$$

We have the following.

**Theorem S.5.** *Let* $R$ *be either a positive number or* $+\infty$. *Let* $f$ *be a positive function of class* $C^2$ *over* $(0,R)$ *satisfying* (S.5) , (S.6), *and* (S.7). *Suppose, moreover, that for some* $0 < r_0 < R$, $f$ *is of class* $C^4$ *over* $[0,r_0)$ *and of class* $C^5$ *over* $(0,r_0)$, *and that* $f^{(5)}$ *is bounded in* $(0,r_0)$. *Then* $U$ *is a solution to* (S.1), *with* $\mathcal{E}$ *given by* (S.2), *of class* $C^1$ *over* $D(R)$ *and of class* $C^2$ *over* $D(R)\backslash\{(0,0)\}$.

It is interesting to consider whether or not the solution $U$ is of class $C^2$ over the entirety of disc $D(R)$. The following theorem specifies certain conditions on the function $f$ that must be met for the answer to be in the affirmative.

**Theorem S.6.** *Let $R$ be either a positive number or $+\infty$. Let $f$ be a positive function of class $C^2$ over $(0, R)$ and, for some $0 < r_0 < R$, of class $C^4$ over $[0, r_0)$ satisfying* (S.5), (S.6), *and* (S.7). *Suppose that $U$ is of class $C^2$ over $D(R)$. Then $f'''(0) = f^{(4)}(0) = 0$.*

We conclude this section with a simple sufficient condition for $U$ to be of class $C^2$ over $D(R)$.

**Theorem S.7.** *Let $R$ be either a positive number or $+\infty$. Let $f$ be a positive function that is of class $C^2$ over $(0, R)$, satisfies* (S.7), *and, for some $0 < r_0 < R$, $f(r) = r^2$ whenever $0 \le r < r_0$. Then $U$ is of class $C^2$ over $D(R)$.*

*S.3.2. Refinements*

We now specify certain classes of functions $f$ to which the results of the previous section are applicable. One of these classes will be used to generate a counter-example to Bruss' assertion mentioned in the introduction to this supplement.

**Theorem S.8.** *Let $R$ be a positive number. Let $g : (0, R) \to [0, 1)$ be a function of class $C^2$ such that $g'$ and $g''$ are non-negative, $g'$ is bounded in $(0, r_0)$ for some $0 < r_0 < R$, and $\lim_{r \to 0} g(r) = 0$. Then the function $f$ defined by*

$$f(r) = \frac{r^2}{1 - g(r)} \quad (0 < r < R) \tag{S.8}$$

*is of class $C^2$ and satisfies* (S.5), (S.6), *and* (S.7).

Notice that if we let $R = 1$ and $g(r) = r^2$ for $0 < r < 1$, then the function $f$ given by (S.8), namely $r^2(1 - r^2)^{-1}$, corresponds to the image of the unit hemisphere. Let $U$ be the corresponding (non-circularly-symmetric) solution to (S.1) with $\mathcal{E}$ as in (S.2). Since $f^{(4)}(0) = 1$, it follows from Theorem S.6 that $U$ is not of class $C^2$. Of course, this result can independently be inferred from uniqueness results, mentioned in the introduction to the supplement, due to Deift and Sylvester, and Brooks.

Let $R$ be a positive number. Let $r_0$ and $r_1$ be such that $0 < r_0 < r_1 < R$. Let $\varphi : (0, R) \to [0, 1]$ be a continuous function vanishing on $(0, r_0]$ and equal to 1 on $[r_1, R)$. For each $0 < r < R$, set

$$g(r) = c \int_0^r \varphi(x)(r - x)\, dx,$$

where

$$c = \left[ \int_0^R \varphi(x)(R - x)\, dx \right]^{-1}.$$

Clearly, $g$ is of class $C^2$ and, for each $0 < r < R$,

$$g'(r) = c \int_0^r \varphi(x)\, dx$$

and $g''(r) = c\varphi(r)$. Accordingly, $g$ meets the conditions specified in Theorem S.8. Let $f$ be the function given by (S.8) and $U$ be the corresponding solution to (S.1) in which $\mathcal{E}$ is given by (S.2). Then, $\lim_{r \to R} g(r) = 1$ and so $\lim_{r \to R} f(r) = +\infty$. Since $g$ vanishes on $(0, r_0)$, it follows that $f(r) = r^2$ for $0 < r \le r_0$. Thus, by Theorem S.7, $U$ is of class $C^2$ over $D(R)$. A straightforward computation shows that $\int_0^R \sqrt{f(r)}\, dr < +\infty$. When combined with Theorem S.2, this relation implies that $U$ is bounded.

It is now clear that our goal expressed in the introduction to the supplement is achieved: the pair $(f, U)$ provides a desired counter-example to Bruss' assertion.

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