# Complete Axiomatizations for Logics of Knowledge and Past Time
# (Long Version)

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**Abstract**

Sound and complete axiomatizations are provided for two different logics involving modalities for knowledge and both past and future time modalities. The logics considered allow for multiple agents with unique initial state and synchrony. Such semantic restrictions are of particular interest in the context of past time modalities since both synchrony and unique initial state restrictions are not expressible without past time modalities. The synchrony restriction gives every agent access to a system clock.

Keywords: Combinations of Modal Logics, Proof Theory.

## 1 Introduction

There has been significant interest in multi-modal logics combining operators for knowledge and time in recent years [5, 3, 7]. With only a few exceptions [3], this literature deals with *future time* temporal operators. In this paper we consider the effect of adding past time operators to such logics.

There are some compelling reasons to consider this extension. One of the topics of interest in the literature has been the interaction between knowledge and time when a variety of semantic properties are assumed, such as uniqueness of initial states, synchrony, perfect recall and no learning (a dual of perfect recall). These properties lead to interaction axioms, which involve both epistemic and temporal operators. Halpern, van der Meyden and Vardi [7] provide complete axiomatizations for all the axiomatizable cases arising out of combinations of these assumptions. However, their results indicate that in some cases, some of the properties have no impact on the axiomatization. For example, [7] obtains identical complete axiomatizations in for the cases of no assumptions, synchrony alone, unique initial states alone, and for both synchrony and unique initial states. This indicates that the logic with future time axioms is too weak to fully express the properties of unique initial states and synchrony. It has also been noted that past time operators allow for a much cleaner axiom for perfect recall in an asynchronous setting [9].

Another reason to consider knowledge in combination with past time operators is that *knowledge-based programs* [2] are better behaved with past-time operators than with future time operators. A knowledge-based program is like a standard program with formulas expressing the knowledge of the agent allowed to occur as conditions in conditional statements. A concrete implementation of such a program replaces the knowledge conditions by concrete conditions of the agent's local state. Knowledge-based programs behave somewhat like specifications, and in general, may have zero, one, or many different implementations. However, it is possible to provide conditions under which there is guaranteed to be a unique implementation [2]. One of these conditions is when the system is synchronous, and all knowledge tests involve only past time operators.

We would like to have an interaction axiom for each of the properties mentioned above, such that combinations of properties can be handled by combining their corresponding axiom. In this paper, we take a step in this direction by providing axioms which individually characterize the properties of unique initial states and synchrony. (We will deal with combinations in future work.) As already remarked, a past time axiom for perfect recall is already given in [9]. The property of no learning is best captured by the future time axiom in [7].

The synchrony restriction is particularly interesting since the axiomatization appears to require a complex automaton-based rule. We give the rather interesting completeness proof here.

In section 2 we give a concise definition of the logics we study. In section 3, we will describe the set of axioms and rules. In sections 4 and 5 we show that the axiomatization for systems with unique initial states is sound and complete. In sections 6 and 7 we show that the axiomatization for synchronous systems is sound and complete. Future work will be indicated in section 8.

## 2 Semantics

The language is given by the abstract syntax:

$$\alpha = x \mid \neg\alpha \mid \alpha \wedge \alpha' \mid \bigcirc\alpha \mid \alpha\, U\, \alpha' \mid \textcircled{w}\alpha \mid \alpha\ \mathcal{S}\ \alpha' \mid K_i\alpha$$

where $x \in \mathcal{V}$ is some propositional atom, and $1 \leq i \leq k$ is the index of an agent. The operators are respectively *not, and, tomorrow, until, weak yesterday, since and i-knows*, and have their usual meaning. Along with the usual propositional abbreviations (*true*, *false*, $\vee$, $\rightarrow$) we will also use the temporal abbreviations: $\ominus\alpha = \neg\textcircled{w}\neg\alpha$; $\Diamond\,\alpha = \mathit{true}\, U\, \alpha$; $\diamondminus\,\alpha = \mathit{true}\ \mathcal{S}\ \alpha$; $\Box\alpha = \neg\Diamond\neg\alpha$; and $\boxminus\alpha = \neg\diamondminus\neg\alpha$, and the epistemic operator, $L_i\alpha = \neg K_i\neg\alpha$.

For the semantics, we suppose a model is given by a set of runs, and each formula is evaluated with respect to some time in some run. Formally, a model is given by:

$$M \subseteq \{r \mid r : \omega \longrightarrow \wp(\mathcal{V}) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_k\} = \mathcal{R},$$

where $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are the local states of each agent. The semantics are given with respect to one run $r \in M$ and one moment of time, $n \in \omega$. We inductively define $M, r, n \models \alpha$ as follows:

$$M, r, n \models x \iff x \in r(n)_0 \tag{1}$$
$$M, r, n \models \neg\alpha \iff M, r, n \not\models \alpha \tag{2}$$
$$M, r, n \models \alpha \wedge \alpha' \iff M, r, n \models \alpha \text{ and } M, r, n \models \alpha' \tag{3}$$
$$M, r, n \models \bigcirc\alpha \iff M, r, n+1 \models \alpha \tag{4}$$
$$M, r, n \models \alpha\, U\, \alpha' \iff \exists m \geq n,\ M, r, m \models \alpha' \text{ and } n \leq j < m \Rightarrow M, r, j \models \alpha \tag{5}$$
$$M, r, n \models \textcircled{w}\alpha \iff n = 0 \text{ or } M, r, n-1 \models \alpha \tag{6}$$
$$M, r, n \models \alpha\ \mathcal{S}\ \alpha' \iff \exists m \leq n,\ M, r, m \models \alpha' \text{ and } m < j \leq n \Rightarrow M, r, j \models \alpha \tag{7}$$
$$M, r, n \models K_i\alpha \iff M, r', m \models \alpha\ \forall r' \in M, \forall n \in \omega \text{ where } r(n)_i = r'(m)_i \tag{8}$$

for each agent $i$.

This gives the most general description of a language that describes knowledge and past time. However there are several useful restrictions we will consider:

- We say a model has a *unique initial state* if for all $r, r' \in M$, $r(0)_i = r'(0)_i$, for all $i \in \{1, \ldots, k\}$;
- We say a model is *synchronized* if for all $r, r' \in M$, for all $n, m \in \omega$, $r(n)_i = r'(m)_i \implies n = m$, for all $i \in \{1, \ldots, k\}$;

There are several other semantic restrictions that can be applied to combinations of temporal and modal logic, including *perfect recall* and *no learning.* We have chosen to focus on synchronization and unique initial state restrictions in this paper as they are especially relevant to temporal logics with past. The synchronization and unique initial state restrictions have little effect in logics without past operators, as these restrictions do not alter the set of valid formulas.

Once past operators are added to the language, the synchronization restriction has a dramatic affect on the set of valid formulas. Since every agent *knows* the time, an axiomatization must allow reasoning about which formulas can be true at which times. For example, if there is some formula, $\alpha$, that is true at only even times, then if an agents even suspects that $\alpha$ might be true at some time, then that agent should know that every formula that is true at only odd times must be false. This situation is captured in the following formula, which is a validity in the synchronized semantics.

$$L_i(x \wedge \boxminus(x \leftrightarrow \textcircled{w}\neg x)) \rightarrow K_i(\boxminus(y \leftrightarrow \textcircled{w}\neg y) \rightarrow y) \tag{9}$$

# 3 Axioms

In this section, we describe the axioms and inference rules that we need for reasoning about knowledge and time for various classes of systems, and state the completeness results.

For reasoning about knowledge alone, the following system, with axioms K1–K5 and rules of inference R1–R2, is well known to be sound and complete [1, 6]:

K1. All tautologies of propositional logic
K2. $K_i\varphi \wedge K_i(\varphi \rightarrow \psi) \rightarrow K_i\psi,\ i = 1, \ldots, k$
K3. $K_i\varphi \rightarrow \varphi$, $i = 1, \ldots, k$
K4. $K_i\varphi \rightarrow K_iK_i\varphi,\ i = 1, \ldots, k$
K5. $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi,\ i = 1, \ldots, k$
R1. From $\varphi$ and $\varphi \rightarrow \psi$ infer $\psi$
R2. From $\varphi$ infer $K_i\varphi,\ i = 1, \ldots, k$

This axiom system is known as S$5_m$.

For reasoning about the temporal operators individually, the following system (together with K1 and R1), can be shown to be sound and complete [11]:

F1. $\bigcirc(\varphi \rightarrow \psi) \rightarrow \bigcirc\varphi \rightarrow \bigcirc\psi$
F2. $\bigcirc(\neg\varphi) \leftrightarrow \neg\bigcirc\varphi$
F3. $\varphi\, U\, \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi\, U\, \psi))$
FP. $\varphi \rightarrow \bigcirc\ominus\varphi$
PF. $\varphi \rightarrow \textcircled{w}\bigcirc\varphi$
RT1. From $\varphi$ infer $\bigcirc\varphi$
RP1. From $\varphi$ infer $\textcircled{w}\varphi$.

P1. $\textcircled{w}(\varphi \rightarrow \psi) \rightarrow \textcircled{w}\varphi \rightarrow \textcircled{w}\psi$
P2. $\ominus\neg\varphi \rightarrow \neg\ominus\varphi$
P3. $\varphi\ \mathcal{S}\ \psi \leftrightarrow \psi \vee (\varphi \wedge \ominus(\varphi\ \mathcal{S}\ \psi))$
P4. $\mathit{true}\ \mathcal{S}\ \textcircled{w}\mathit{false}$
RT2. From $\varphi' \rightarrow \neg\psi \wedge \bigcirc\varphi'$ infer $\varphi' \rightarrow \neg(\varphi\, U\, \psi)$
RP2. From $\varphi' \rightarrow \neg\psi \wedge \textcircled{w}\varphi'$ infer $\varphi' \rightarrow \neg(\varphi\ \mathcal{S}\ \psi)$

This set of axioms gives a sufficient axiomatization of knowledge with past time. To allow for the unique initial states restriction, we add the following axiom:

$$\text{UIS.}\quad \boxminus(\textcircled{w}\mathit{false} \rightarrow K_i\alpha) \rightarrow K_j \boxminus(\textcircled{w}\mathit{false} \rightarrow \alpha),\ \ i, j = 1, \ldots k.$$

For the rules required for synchronized time, we must first define a characteristic formula for a transducer (or deterministic linear automaton over a one letter alphabet).

**Definition 3.1:** Let $X$ be a set of propositional atoms $X$, where $a_0 \subseteq X$, and for every $a \subseteq X$, let $a' \subseteq X$ be defined. For each $a \subseteq X$, we let $\overline{a} = \bigwedge_{x \in a} x \wedge \neg \bigvee_{x \in X \setminus a} x$. For any such set $X$, and function $'$, we define

$$\alpha = \diamondminus \left( \textcircled{w}\mathit{false} \wedge \overline{a_0} \wedge \Box \bigwedge_{a \subset X} (\overline{a} \rightarrow \bigcirc\overline{a'}) \right).$$

to be a *characteristic formula*

It should be clear to see that a characteristic formula is always satisfiable. It simply declares which atoms should be true at which times in a deterministic manner. Let $var(\alpha)$ be the set of propositional atoms appearing in a formula, $\alpha$.

In the case of synchronized time we require two new rules.

AUT. From $\chi \rightarrow \beta$ infer $\beta$, where $var(\chi) \cap var(\beta) = \emptyset$ and $\chi$ is a characteristic formula.

SYNC. From $\alpha \rightarrow \beta$ infer $\alpha \rightarrow K_i\beta$, where $var(\alpha) \cap var(\beta) = \emptyset$, $i = 1, \ldots, k$.

The rule, AUT, is interesting in that it does not use any knowledge operators. Such a rule is valid in temporal logics with past but has rarely been used in proof systems (see for example, the AA rule of [10]). We require the rule to add extra propositions into a proof when the propositions already in the proof do not yield sufficient information about the system clock.

While the rule, AUT, provides sufficient atoms to be able to compare histories, the rule, SYNC, is the mechanism by which we do the comparison. For an example of the effectiveness of the rule, consider the formula (9). Since it is clear that

$$\vdash (x \wedge \boxminus(x \leftrightarrow \textcircled{w}\neg x)) \rightarrow (\boxminus(y \leftrightarrow \textcircled{w}\neg y) \rightarrow y), \tag{10}$$

by the completeness of the temporal rules and axioms, the provability of (9) follows directly from the SYNC rule (and some epistemic axioms and rules).

## 4 Soundness for unique initial states

Suppose the axiom was not sound. Then there would be some model $M$, such that for some $r \in M$ and some $j$, $M, r, j \models \boxminus(\textcircled{w}\mathit{false} \rightarrow K_i\alpha) \wedge \neg K_i \boxminus(\textcircled{w}\mathit{false} \rightarrow \alpha)$. Therefore there must be some $r' \in M$ such that $r(j)_i = r'(j)_i$ such that $M, r', j \models \neg \boxminus(\textcircled{w}\mathit{false} \rightarrow \alpha)$. Thus $M, r', 0 \models \neg\alpha$, and $M, r, 0 \models K_i\alpha$ contradicting the unique initial states requirement of the model.

## 5 Completeness for unique initial states

To prove the axiom system augmented with UIS is complete we use a standard Henkin-style construction with finite sets of formulas. Given a consistent formula, $\psi$, we show that $\psi$ has a model generated from the maximal consistent subsets of some closure set (see, for example [4]). We define the closure set in two stages. Given $\psi$, let $\Gamma_\psi = \{\alpha, \neg\alpha, \textcircled{w}\mathit{false} |\ \alpha \subseteq \psi\}$. As usual we let $\Sigma$ be the set of maximally consistent sets of formulas, and $S_\psi = \Sigma \cap \Gamma_\psi$. We let $S^0_\psi = \{s \in S_\psi \mid \textcircled{w}\mathit{false} \in s\}$.

For the next stage, we let $\Gamma = \Gamma_\psi \cup \{\Diamond(\textcircled{w}\mathit{false} \wedge \hat{s}) \mid s \in S^0_\psi\}$ where $\hat{s}$ is the conjunction of the formulas in $s$. We define $S = \Sigma \cap \Gamma$ and define the relations $\leadsto, \sim_i \subseteq S \times S$ as:

- $s \leadsto t$ if and only if there exists $\Delta, \Delta' \in \Sigma$ such that $s = \Delta \cap \Gamma_\psi$, $t = \Delta' \cap \Gamma_\psi$ and for all $\alpha \in \Delta'$, $\bigcirc\alpha \in \Delta$;
- $s \sim_i t$ if and only if there exists $\Delta, \Delta' \in \Sigma$ such that $s = \Delta \cap \Gamma_\psi$, $t = \Delta' \cap \Gamma_\psi$ and for all $K_i\alpha \in \Delta$, $K_i\alpha \in \Delta'$.

Note that $\sim_i$ is an equivalence relation, and for all $s \in S$, we let $[s]_i$ be the corresponding equivalence class. We let

$$R = \{r : \omega \rightarrow S \mid \forall i, r(i) \leadsto r(i+1) \text{ and } \alpha\, U\, \beta \in r(i) \Rightarrow \exists d \geq i, \beta \in r(d)\}.$$

From $R$ we can derive a model $M = \{\pi_r : \omega \rightarrow \wp(\mathcal{V}) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_k \mid r \in R\}$ where $\pi_r(j)_0 = r(j) \cap \mathcal{V}$, and $\pi_r(j)_i = [r(j)]_i$. Finally, for every $r \in R$ we let $M_r \subset M$ be defined to be the smallest set such that $\pi_r \in M_r$, and for every $\pi_t \in M_r$, $\{\pi_u \in M \mid \exists i, j \text{ s.t. } t(j) \sim_i u(j)\} \subseteq M_r$.

The standard approach here is to extend $\psi$ to a maximal consistent set and use this to find a run $r$ with a state containing $\psi$. We then prove a truth lemma on $M_r$, i.e. for every $j$ we show $\alpha \in r(j)$ if and only if $M, \pi_r, j \models \alpha$. Therefore to complete the proof all we have to do is show that the resulting model satisfies the unique initial states constraint. We use the following tautology:

**Lemma 5.1:** *For all $s \in S$,*

$$\vdash \textcircled{w}\mathit{false} \rightarrow (\hat{s} \rightarrow \Box(K_i \boxminus(\textcircled{w}\mathit{false} \rightarrow \neg K_j \neg\hat{s})))$$

**Proof:** Let

$$\gamma = \textcircled{w}\mathit{false} \wedge (\hat{s} \wedge \Diamond(L_i \Diamond(\textcircled{w}\mathit{false} \wedge K_j \neg\hat{s})))$$

By taking the contrapositive of UIS we have $\vdash \neg K_i \neg \diamondminus (\textcircled{w}\mathit{false} \wedge \neg\alpha) \rightarrow \diamondminus (\textcircled{w}\mathit{false} \wedge \neg K_j \alpha)$. Let $\alpha = \neg K_j \neg \hat{s}$. Applied to $\gamma$ we have

$$
\begin{array}{rll}
UIS & \vdash \gamma \rightarrow (\textcircled{w}\mathit{false} \wedge \hat{s} \wedge \Diamond \diamondminus (\textcircled{w}\mathit{false} \wedge \neg K_j \neg K_j \neg \hat{s})) & (11) \\
K5 & \vdash \neg K_j \neg \hat{s} \rightarrow K_j \neg K_j \neg \hat{s} & (12) \\
K1 & \vdash \neg K_j \neg K_j \neg \hat{s} \rightarrow K_j \neg \hat{s} & (13) \\
LTL & \vdash \gamma \rightarrow (\textcircled{w}\mathit{false} \wedge \hat{s} \wedge \Diamond \diamondminus (\textcircled{w}\mathit{false} \wedge K_j \neg \hat{s})) & (14) \\
K1 & \vdash \gamma \rightarrow \hat{s} \wedge \neg \hat{s} & (15) \\
K1 & \vdash \neg \gamma & (16)
\end{array}
$$

Since $\neg\gamma$ is equivalent to $\textcircled{w}\mathit{false} \rightarrow (\hat{s} \rightarrow \Box(K_i \boxminus (\textcircled{w}\mathit{false} \rightarrow \neg K_j \neg \hat{s})))$, the proof is complete. ▮

**Corollary 5.2:** *The model $M_r$ satisfies the unique initial states constraint.*

**Proof:** If this were not true there would be some runs with non-unique initial states. Thus there would be some $s(0), t(0), s(u), t(v) \in S$ (where $s, t \in R$) such that $s(u) \sim_i t(v)$, but $s(0) \not\sim_j t(0)$. However, by the above lemma,

$$\vdash \widehat{s(0)} \rightarrow \Box K_i \boxminus (\textcircled{w}\mathit{false} \rightarrow L_j \widehat{s(0)}).$$

Given the closure $\Gamma$, we know $\vdash \widehat{s(u)} \rightarrow \diamondminus (\textcircled{w}\mathit{false} \wedge \widehat{s(0)}))$. Therefore $\vdash \widehat{s(u)} \rightarrow K_i \boxminus (\textcircled{w}\mathit{false} \rightarrow L_j \widehat{s(0)})$, and since $s(u) \sim_i t(v)$ we must have $\vdash \widehat{t(v)} \rightarrow \boxminus (\textcircled{w}\mathit{false} \rightarrow L_j \widehat{s(0)})$. Therefore it follows that $t(\hat{0}) \wedge L_j s(\hat{0})$ must be consistent, contradicting the assumption $s(0) \not\sim_j t(0)$. ▮

## 6 Soundness for synchronized time

The soundness of the rule AUT is straightforward, and is left to the reader. To show SYNC is sound, suppose that $\alpha$ and $\beta$ do not share propositional atoms, $\alpha \rightarrow \beta$ is a validity, but $\alpha \wedge L_i \neg \beta$ has some model, $M$. Therefore there are runs $r, s \in M$ and some $j$ such that $M, r, j \models \alpha$, and $M, s, j \models \neg\beta$, and $r(j)_i = s(j)_i$. Note that the interpretation of $\alpha$, and the interpretation of $\beta$ can only depend on the propositional atoms that appear in $\alpha$ or $\beta$ (this can be seen by the recursive definition of the $\models$ relation).

Now let $M^+$ be a new model defined by $M^+ = \{r_j^i | r_i, r_j \in M\}$, where the run $r_j^i$ is defined by $r_i^j(u) = (a, l_1, \ldots, l_k)$ where

- $a = (r_i(u)_0 \cap var(\alpha)) \cup (r_j(u)_0 \cap var(\beta))$
- $l_m = (r_i(u)_m, r_j(u)_m)$

Note that this requires that the runs are synchronized.

We can show that $M, r_i, u \models \alpha$ if and only if $M^+, r_i^j, u \models \alpha$ for all $j$, and $M, r_i, u \models \beta$ if and only if $M^+, r_j^i, u \models \beta$ for all $j$. (This is done by induction over the complexity of formulas, using the semantic descriptions given, and is left to the reader). If we let $r = r_a$ and $s = r_b$, it follows that $M^+, r_a^b, j \models \alpha \wedge \neg\beta$, contradicting the fact that $\alpha \rightarrow \beta$ is a validity.

# 7 Completeness for synchronized time

This proof of completeness will also use a Henkin style construction. Given a consistent formula, $\psi$, we will use a set of maximal consistent subsets to construct a model of $\psi$. We use the strategy used in [7] to construct the model as a series of levels, where each level is represented by a string of agent indexes. We let $\lambda$ refer to the empty string, $\tau i$ be the string $\tau$, concatenated with the index $i$, and $\tau \backslash i$ be the largest string $\mu$ such that $\mu i$ is a prefix of $\tau$, or $\lambda$ if such a String does not exist. We also use $<$, $\leq$ as relations where $\tau \leq \sigma$ ($\tau < \sigma$) indicates that $\tau$ is a (proper) prefix of $\sigma$.

We define the following hierarchy of languages: We let $\mathcal{L}$ be the language defined above (for $k$ agents), and define the hierarchy over sequences of agents (the nestings of knowledge operators).

1. $\mathcal{L}_\lambda = \{\alpha \in \mathcal{L} \mid \forall \beta \in \mathcal{L}, \forall i \; K_i\beta \nsubseteq \alpha\}$.
2. $\mathcal{L}_{\tau i} = \{\alpha \in \mathcal{L} \mid K_j\beta \subseteq \alpha \Rightarrow$ either $j = i$ and $\beta \in \mathcal{L}_\tau$ or $K_j\beta \in \mathcal{L}_\tau\}$.

We can see that $\mathcal{L}_\lambda$ is the set of all pure temporal formulas, and let $\sigma$ be the smallest string such that $\psi \in \mathcal{L}_\sigma$.

We will now define the *closure* of a formula, $\psi$.

**Definition 7.1:** Given a formula, $\psi$, we let $\Gamma_\psi$ be the *closure* of $\psi$ where $\Gamma_\psi$ is recursively defined such that:

- $\psi \in \Gamma_\psi$.
- $\alpha \subseteq \varphi$ implies $\alpha, \neg\alpha \in \Gamma_\psi$
- $\alpha \in \Gamma_\psi$ implies $\neg K_i\alpha \in \Gamma_\psi$ and $K_i\alpha \in \Gamma_\psi$ for $i = 1, \ldots, m$.

We also define the *$\tau$-closure* of $\psi$ to be $\Gamma_\psi^\tau = \Gamma_\psi \cap \mathcal{L}_\tau$.

We can now construct a set of maximally consistent subsets of a closure set as follows. Let $\Sigma$ be the set of *maximally consistent sets* of formulas taken from the language (with respect to the axioms given and the two rules for synchronization), and given a set $X$ of formulas, we let $S_X$ be the set of maximally consistent subsets of $X$, (ie $S_X = \{\Delta \cap X \mid \Delta \in \Sigma\}$).

We then define the temporal relation $\rightsquigarrow \subseteq S_X \times S_X$ (where $X$ will be assumed from context) by $s \rightsquigarrow t$ if and only if $\bigwedge_{\alpha \in s} \alpha \wedge \bigcirc \bigwedge_{\alpha \in t} \alpha$ is consistent. The knowledge relations are quite complex, and will be constructed using the following definitions and lemmas.

The structure of the proof will be as follows. We will that any consistent formula, $\psi$, has a model. We will show this by creating a model of $\psi$ using the maximal consistent subsets of the closure of $\psi$, along with with some additional information. To enforce the synchronization constraint, we will construct the model by induction over the knowledge depth of subformulas of $\psi$. We will then prove the correctness of the construction by induction over the complexity of formulas.

The inductive construction of the model will require the following definitions. These constructions are given so that if we are considering formulas of knowledge sequence $\tau i$, then the closure includes an additional formula $\chi_\tau$ that describes a model for subformulas in the closure of knowledge sequence $\tau$. We do this by induction, where the base case is

$$X_\lambda = \Gamma_\psi^\lambda \cup \Gamma^\lambda_{\Diamond\!\!\!\!\!-\;\textcircled{w} \mathit{false}}.$$

Given $X_\tau$, for any $\tau$, we can then define $S_\tau$ (the maximally consistent subsets of $X_\tau$, $A_\tau$ (a transducer showing which subsets are consistent with which times), $\chi_\tau$ (the characteristic formula of the transducer), and $X_{\tau i}$ (the inductive step). This is done as follows:

- $S_\tau = S_{X_\tau}$.
- For all $\tau$, given $S_\tau$ and $\rightsquigarrow$ (defined above) we let $A_\tau$ be a transducer given by the tuple $(Q_\tau, p_\tau, \delta_\tau)$ where:
  - $Q_\tau = \wp(S_\tau)$ is the set of states;
  - $p_\tau = \{s \in S_\tau \mid \textcircled{w}\mathit{false} \in s\}$
  - $\delta_\tau : Q_\tau \to Q_\tau$ is the transition function defined by $\delta_\tau(q) = \{t \mid \exists s \in q,\ s \rightsquigarrow t\}$.

  This transducer is defined to identify states which are reachable in the constructed model at a given time. The run of $A_\tau$ is the sequence from $Q_\tau$, $(p_\tau, \delta_\tau(p_\tau), \delta_\tau^2(p_\tau), \ldots)$.
- $\chi_\tau$ is the characteristic formula of $A_\tau$. To define $\chi_\tau$, for each $s \in S_\tau$, let $x_s$ be a propositional atom not appearing in $\Gamma_\tau$, and for all $q \in Q_\tau$, let $\overline{q} = \bigwedge_{s\in q} x_s \wedge \neg \bigvee_{s \notin q} x_s$. Then

$$\chi_\tau = \Diamond\left(\textcircled{w}\mathit{false} \wedge \overline{p_\tau} \wedge \Box \bigwedge_{q\in Q_\tau} (\overline{q} \to \bigcirc\overline{\delta_\tau(q)})\right).$$

- $X_{\tau i} = \Gamma_\psi^{\tau i} \cup \Gamma_{\chi_\tau}^{\lambda}$.

The proof of completeness will follow from the following lemmas. The first three are technical lemmas which contribute to the proof of the fourth.

**Lemma 7.2:** *For all* $\tau$, $\vdash \chi_\tau \wedge \overline{q} \to K_i \bigvee_{s\in q} \hat{s}$.

**Proof:** By the construction the transducer $A_\tau$ the initial state $p_\tau$ is the set of all maximal consistent subsets which are consistent with $\textcircled{w}\mathit{false}$, so the following is a validity

$$\vdash \textcircled{w}\mathit{false} \wedge \chi_\tau \to \bigvee_{s\in p_\tau} \hat{s}. \tag{17}$$

The transition function $\delta_\tau$ is defined to map a set, $q$, of maximal consistent subsets to the set of all maximal consistent subsets that are consistent with $\textcircled{w} \bigvee_{s\in q} \hat{s}$. Therefore the following is a tautology:

$$\vdash \chi_\tau \wedge \left(\bigwedge_{q\in Q_\tau}\left(\overline{q} \to \bigvee_{s\in q} \hat{s}\right)\right) \to \bigcirc\left(\chi_\tau \wedge \left(\bigwedge_{q\in Q_\tau}\left(\overline{q} \to \bigvee_{s\in q} \hat{s}\right)\right)\right) \tag{18}$$

Applying temporal validities to these formulas we can derive

$$\vdash \chi_\tau \wedge \overline{q} \to \bigvee_{s\in q} \hat{s} \tag{19}$$

Since the propositional atoms in $\chi_\tau$ are defined to be disjoint from those in $X_\tau$ the result follows from the Sync rule. ▮

The following lemma allows us to apply the previous lemma at different levels of knowledge depth.

**Lemma 7.3:** *For all $\tau$, $j \in \omega$ and $i \leq m$,* $\vdash \chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \to \Big(\chi_{\tau\backslash i} \to \overline{\delta^j_{\tau\backslash i}(p_{\tau\backslash i})}\Big)$

**Proof:** We will prove this by induction. We will show

$$\vdash \chi_{tau} \wedge \overline{\delta^j_\tau(p_\tau)} \to \Big(\chi_{\tau^-} \to \overline{\delta^j_{\tau^-}(p_{\tau^-})}\Big) \tag{20}$$

and since we can asuume that the propositional atoms in $\chi_\tau$, and $\chi_{\tau^-}$ are disjoint, the result follows by the rule, Aut. Since (20) is a pure temporal formula, it is sufficient for us to show it is valid using semantic reasoning, since all temporal validities are provable in the proof system. To do this we simply have to examine the construction of $A_\tau$. It is a transducer defined over a finite set of states. Therefore the run of $A_\tau$ must have the form

$$q_0, \dots, q_x, \dots, q_y, q_x, \dots, q_y, q_x, \dots,$$

where $0 < x \leq y$, and each state has a unique index throughout the run. We let $\Pi_\tau = 1 + y - x$ be the *period* of $A_\tau$. Likewise the run of $A_{\tau^-}$ must have the form $p_0, \dots, p_w, \dots, p_z, p_w, \dots$.

By definition the states of $A_\tau$ are maximally consistent subsets of $X_\tau$, where $\chi_{\tau^-} \in X_\tau$. Since $\chi_{\tau^-}$ is always consistent (by the soundness of Aut), for all $n \leq y$, there is some $a \in q_n$ such that $\chi_{\tau^-} \in a$. Therefore for all $n < x$,

$$\vdash \chi_\tau \wedge \overline{q_n} \to \Big(\chi_{\tau^-} \to \overline{\delta^n_{\tau^-}(p_{\tau^-})}\Big)$$

since $\chi_\tau \wedge \overline{q_n}$ can only be consistent with time $n$. As $\chi_\tau \wedge \overline{q_{x-1}}$ can only be consistent with time $x-1$ it follows that for all $a \in q_x$, $\chi_{\tau^-} \in a$ implies $\overline{\delta^x_{\tau\backslash i}(p_{\tau\backslash 1})} \in a$. The only way this can occur is if $\Pi_{\tau^-}$ divides $\Pi_\tau$, and consequently for all $n \leq y$ there is a unique $n' \leq z$ such that $\vdash \chi_\tau \wedge \overline{q_n} \to (\chi_{\tau^-} \to \overline{p_{n'}})$. Therefore (20) follows from the completeness of the temporal axioms and rules, and the lemma follows by induction, and the rule, Aut. ▮

We will now restrict our attention to sets $s \in S_{\tau i}$, such that $\chi_\tau \in s$. By the construction of the set $S_{\tau i}$ and the rule, AUT, every consistent formula in $\Gamma^{\tau i}_\psi$ must be an element of some set in $S_{\tau i}$. Let $T_\lambda = S_\lambda$ and $T_{\tau i} = \{s \in S_{\tau i} \mid \chi_\tau \in s\}$. Given any set, $t$ of formulas, we let $t^i = \{\alpha \mid K_i\alpha \in t\}$. We require some additional definitions to allow us to compare maximal consistent subsets at different levels. If $\tau = \mu j$, we let $\tau^- = \mu$.

**Definition 7.4:** For all $\tau \neq \lambda$, we define the relation $\prec_i \subset T_{\tau\backslash i} \times T_\tau$ and say $t \prec_i s$ ($t$ *$i$-supports* $s$) if

- For some $q \in Q_{\tau^-}$, $\overline{q} \in s$ and there is some $a \in q$ such that $t \subseteq a$.
- $t^i \subseteq s^i \subseteq t$.

This definition is constructed such that for all $s \in T_\tau$, and for all $t \in T_{\tau\backslash i}$, $t \prec_i s$ if and only if $\hat{s} \wedge L_i\hat{t}$ is consistent. To use this property we require the following lemma.

**Lemma 7.5:** *For all $\tau$, for all $j$, for all $s \in \delta^j_\tau(p_\tau) \cap T_\tau$, if $L_i\gamma \in s$, then there is some $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i}) \cap T_{\tau\backslash i}$ such that $\gamma \in t$ and $t \prec_i s$.*

**Proof:** Suppose for contradition that there exists $\tau$, $j \in \omega$, $s \in \delta^j_\tau(p_\tau) \cap T_\tau$ and some $L_i\gamma \in s$ such that for all $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i}) \cap T_{\tau\backslash i}$, if $t \prec_i s$, then $\gamma \notin t$. We will convert this statement into a formula and use the proof theory to derive a contradiction.

Firstly, we note that for all $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i}) \cap T_{\tau\backslash i}$, $t \not\prec_i s$ implies $\vdash \hat{t} \to K_i\neg\hat{s}$. This follows from the facts: if $t^i \not\subseteq s^i$, then for some $\alpha$, $K_i\alpha \in t$, and $\neg K_i\alpha \in s$; if $s^i \not\subseteq t$, then for some $\alpha$, $K_i\alpha \in s$ and $\neg\alpha \in t$); and $\chi_{\tau\backslash i} \in t$ and $\chi_{\tau^-} \in s$.

Therefore it follows that for all $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i})$, either $t \not\prec_i s$, or $\gamma \notin t$, or $t \notin T_{\tau\backslash i}$. Thus the following is a tautology:

$$\vdash \bigwedge_{t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i})} \left(\hat{t} \to (K_i\neg\hat{s} \vee \neg\gamma \vee \neg\chi_{\tau\backslash i})\right). \tag{21}$$

Since $s \in \delta^j_{\tau}(p_{\tau}) \cap T_{\tau}$ it follows that $\chi_{\tau^-}, \overline{\delta^j_{\tau^-}(p_{\tau^-})} \in s$. By Lemma 7.2 and Lemma 7.3 we know

$$\vdash \hat{s} \to \chi_{\tau^-} \wedge \overline{\delta^j_{\tau^-}(p_{\tau^-})} \tag{22}$$

$$\vdash \chi_{\tau^-} \wedge \overline{\delta^j_{\tau^-}(p_{\tau^-})} \to \left(\chi_{\tau\backslash i} \to \overline{\delta^j_{\tau\backslash i}(p_{\tau\backslash i})}\right) \tag{23}$$

$$\vdash \chi_{\tau\backslash i} \wedge \overline{\delta^j_{\tau\backslash i}(p_{\tau\backslash i})} \to K_i \bigvee_{t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i})} \hat{t} \tag{24}$$

$$\vdash \chi_{\tau\backslash i} \to \left(\hat{s} \to K_i \bigvee_{t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i})} \hat{t}\right) \tag{25}$$

Since either the formula $\chi_{\tau\backslash i}$ is either in $s$ or it shares no propostional atoms with either $X_\tau$ or $X_{\tau\backslash i}$ we can derive (using the Aut rule)

$$\vdash \hat{s} \to K_i \bigvee_{t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i})} \hat{t}. \tag{26}$$

Putting this together with (21) we can show

$$\vdash \hat{s} \to K_i(K_i\neg\hat{s} \vee \neg\gamma \vee \neg\chi_{\tau\backslash i}). \tag{27}$$

It is then a simple matter to show that $s$ is inconsistent, giving us the required contradiction.

$$\begin{array}{rl}
27 & \vdash \hat{s} \to K_i(L_i\hat{s} \to \neg\gamma \vee \neg\chi_{\tau\backslash i}) \\
\mathrm{K3} & \vdash \hat{s} \to K_iL_i\hat{s} \\
\mathrm{K2} & \vdash \hat{s} \to K_i(\gamma \to \neg\chi_{\tau\backslash i}) \\
\mathrm{R2, K2} & \vdash L_i\hat{s} \to K_i(\gamma \to \neg\chi_{\tau\backslash i}) \\
\mathrm{K3} & \vdash L_i\hat{s} \wedge \gamma \to \neg\chi_{\tau\backslash i} \\
\mathrm{Aut} & \vdash L_i\hat{s} \to \neg\gamma \\
 & \vdash \hat{s} \to \neg L_i\gamma.
\end{array}$$

Since $L_i\gamma \in s$ it follows that $s$ is inconsistent. ∎

The above lemma gives us the sufficient machinery to complete the proof. If $\psi$ is consistent, then for some $\sigma$, $\psi$ must belong to $\Gamma^\sigma_\psi$ and $\psi$ must be consistent with $\chi_\tau$, for all $\tau$. Therefore we can find some $s \in T_\sigma$ such that $\psi \in s$. It is clear that the relation $\leadsto$ can be restricted to $T_\sigma$ for all $\sigma$, so we can use this to create a $\sigma$-history (an infinite $\leadsto$-sequence in $T_\sigma$) where all eventualities are satisfied. For every set in this history we can then satisfy any knowledge formulas using Lemma 7.5.

The construction we will use here is given as follows:A *ranked set of height* $\sigma$ is a disjoint union $R = \bigcup_{\tau\leq\sigma} R_\tau$, where for each $\tau \leq \sigma$, $R_\tau$ is a set of labels. For each $r$ in $R_\tau$ we associate a $\tau$-history via a *labeling*, described in the following definition:

**Definition 7.6:** A *labeling*, $\ell$, of a ranked set $R$ of height $\sigma$ is a collection of partial functions $\ell_\tau : R_\tau \times \omega \longrightarrow T_\tau$ for $\tau \leq \sigma$ where:

1. for all $r \in R_\tau$, $\ell_\tau(r, 0) \in p_\tau$;
2. for all $r \in R_\tau$, for all $j \in \omega$, $\ell_\tau(r, j) \leadsto \ell_\tau(r, j + 1)$;
3. for all $r \in R_\tau$, for all $j \in \omega$ for all $\alpha\, U\, \beta \in \ell_\tau(r, j)$ there is some $i \geq j$ such that $\beta \in \ell_\tau(r, i)$.

Hence, for any labeling $\ell$, for any $r \in R_\tau$, $\ell_\tau(r, 0)\ell_\tau(r, 1)\ell_\tau(r, 2)\ldots$ will be a $\tau$-history. The construction must also satisfy all the knowledge formulas. To do this we use the observation that if $L_i\gamma$ appears at some level (say, $L_i\gamma \in \ell_\tau(r, j)$ where $r \in R_\tau$), then a history labeled by an element of $R_{\tau\backslash i}$ is all that is required to satisfy this formula. To facilitate this we use the following definition:

**Definition 7.7:** A *system of support*, $\rho$, for a ranked set $R$ of height $\sigma$ equipped with a labeling $\ell$ consists of, for all $\tau < \sigma$, for all agents $i$, a partial function $\rho^i_\tau : R_{\tau\backslash i} \hookrightarrow R_\tau \times \omega$, such that

1. for all $r \in R_\tau$, for all $j \in \omega$, if $\rho^i_\tau(t) = (r, j)$, then $\ell_{\tau\backslash i}(t, j) \prec_i \ell_\tau(r, j)$.
2. for all $r \in R_\tau$ for all $j \in N$, if $L_i\gamma \in \ell_\tau(r, j)$ then exactly one of the following holds:
   - there is some $t \in R_{\tau\backslash i}$ such that $\rho^i_\tau(t) = (r, j)$ and $\gamma \in \ell_\tau(t, j)$.
   - there is some $t \in R_\mu$ such that $\mu\backslash i = \tau$, and $\rho^i_\mu(r) = (t, j)$.

This gives us enough to define the basic structure.

**Definition 7.8:** Let $\psi \in \Gamma_n$ be a formula, and $\sigma$ an index such that $\psi \in \mathcal{L}_\sigma$. A *$\psi$-Frame* is a triple $(R, \ell, \rho)$ where $R$ is a ranked set of height $\sigma$, $\ell$ is a labeling of $R$ and $\rho$ is a system of support for $R$ and $\ell$, such that for some $r \in R_\sigma$, and some $j \in \omega$, we have $\psi \in \ell_\sigma(r, j)$.

**Lemma 7.9:** *Given any consistent formula, $\psi$, there exists a $\psi$-frame.*

This is left to the reader. The existence of $i$-$\tau$-supports follows from lemma 7.5, and the existence of the $\tau$-labellings follows from the usual reachability arguments.

Given a $\psi$-frame $F$, we can now construct a model, $M_F \subseteq \{\pi : \omega \to \wp(\mathcal{V})\times\mathcal{L}_1\times\ldots\times\mathcal{L}_d\}$ as follows.

- We let the local states for each agent be taken from $R \times \omega$.
- For all $\tau \leq \sigma$, for each $r \in R_\tau$ we define a function $\pi_r : \omega \to \wp(\mathcal{V}) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_d$ by $\pi_r(j) = (a, l_1, \ldots, l_d)$ where:

  - $a = \ell_\tau(r, j) \cap \mathcal{V}$;
  - for each $i = 1, \ldots d$ if $\rho^i_\tau(r) = (t, j)$, then $l_i = \pi_t(j)_i$, and otherwise $l_i = (r, j)$.

It is clear that the model $M_F$ is synchronized.

**Lemma 7.10:** *For all* $\tau \leq \sigma$, $r \in R_\tau$, *for all* $j \in \omega$, *and for all* $\varphi \in \Gamma^\tau$

$$M_F, \pi_r, j \models \varphi \iff \varphi \in \ell_\tau(r, j).$$

**Proof:** This is shown in the usual way, by induction over the complexity of formulas. The inductive steps for all operators are trivial, except the knowledge operators.

- If $\varphi \in \mathcal{V}$, then $M_F, \pi_r, j \models \varphi \iff \varphi \in \ell_\tau(r, j)$, by definition.
- If $\varphi = \alpha \wedge \beta$, then

$$\begin{aligned} M_F, \pi_r, j \models \varphi &\iff M_F, \pi_r, j \models \alpha \text{ and } M_F, \pi_r, j \models \beta \\ &\iff \alpha \in \ell_\tau(r, j) \text{ and } \beta \in \ell_\tau(r, j) \\ &\iff \alpha \wedge \beta \in \ell_\tau(r, j) \end{aligned}$$

- If $\varphi = \neg\alpha$, then

$$\begin{aligned} M_F, \pi_r, j \models \varphi &\iff M_F, \pi_r, j \not\models \alpha \\ &\iff \alpha \notin \ell_\tau(r, j) \\ &\iff \neg\alpha \in \ell_\tau(r, j) \end{aligned}$$

The following four cases use the definition of the relation, $\rightsquigarrow$, and the map, $\ell_\tau$. The complete derivations are left to the reader.

- If $\varphi = \bigcirc\alpha$, then

$$\begin{aligned} M_F, \pi_r, j \models \varphi &\iff M_F, \pi_r, j+1 \models \alpha \\ &\iff \alpha \in \ell_\tau(r, j+1) \\ &\iff \bigcirc\alpha \in \ell_\tau(r, j) \end{aligned}$$

- If $\varphi = \textcircled{w}\alpha$, then

$$\begin{aligned} M_F, \pi_r, j \models \varphi &\iff j = 0 \text{ or } M_F, \pi_r, j-1 \models \alpha \\ &\iff \textcircled{w}\mathit{false} \in \ell_\tau(r, j) \text{ or } \alpha \in \ell_\tau(r, j-1) \\ &\iff \textcircled{w}\alpha \in \ell_\tau(r, j) \end{aligned}$$

- If $\varphi = \alpha\, U\, \beta$, then

$$\begin{aligned} M_F, \pi_r, j \models \varphi &\iff \exists k, \forall i < k, M_F, \pi_r, j+k \models \beta \text{ and } M_F, \pi_r, j+i \models \alpha \\ &\iff \exists k, \forall i < k, \alpha\, U\, \beta \in \ell_\tau(r, j+k) \text{ and } \alpha \in \ell_\tau r, j+i \\ &\iff \alpha\, U\, \beta \in \ell_\tau(r, j) \end{aligned}$$

- If $\varphi = \alpha\ \mathcal{S}\ \beta$, then

$$\begin{aligned} M_F, \pi_r, j \models \varphi &\iff \exists k \leq j, \forall i < k, M_F, \pi_r, j-k \models \beta \text{ and } M_F, \pi_r, j-i \models \alpha \\ &\iff \exists k \leq j, \forall i < k, \alpha\, U\, \beta \in \ell_\tau(r, j-k) \text{ and } \alpha \in \ell_\tau r, j-i \\ &\iff \alpha\ \mathcal{S}\ \beta \in \ell_\tau(r,j) \end{aligned}$$

This leaves the knowledge case, where $\varphi = K_i\alpha$. Suppose that $M_F, \pi_r, j \models \varphi$. Therefore, $M_F, \pi_r, j \models \alpha$ and for all $t$ such that $\pi_r(j)_i = \pi_t(j)_i$ we have $M_F, \pi_t, j \models \alpha$. By the construction of $M_F$, we have two possiblilities:

- For all $t$, such that $\pi_r(j)_i = \pi_t(j)_i$, $t \in R_{\tau\backslash i}$ and $\rho^i_\tau(t) = (r,j)$. By the induction hypothesis, for all such $t$ where $\rho^i_\tau(t) = (r,j)$, $\alpha \in \ell_{\tau\backslash i}(t)$. Suppose for contradiction $\varphi \notin \ell_\tau(r,j)$. Then $L_i\neg\alpha \in \ell_\tau(r,j)$ and by the definition of $\rho^i_\tau$ there would be some $t$ such that $\rho^i_\tau(t) = (r,j)$ and $\alpha \notin \ell_{\tau\backslash i}$, giving us the required contradiction.
- There is some $\mu$ and some $s \in T_\mu$ such that $\mu\backslash i = \tau$ and for all $t$, such that $\pi_r(j)_i = \pi_t(j)_i$, we have $\rho^i_\mu(t) = (s,j)$. By the induction hypothesis, for all such $t$ where $\rho^i_\mu(t) = (s,j)$, $\alpha \in \ell_\tau(t,j)$. Since $\varphi \in \Gamma^{\mu\backslash i}$ we must have $K_i\varphi \in \Gamma^\mu$. Therefore either $K_i\varphi \in \ell_\mu(s,j)$ or $L_i\neg\varphi \in \ell_\mu(s,j)$. In the first case, we have $\ell_\tau(r,j) \prec_i \ell_\mu(s,j)$, so by the definition of $\prec_i$

  $$K_i\varphi \in \ell_\mu(s,j) \Rightarrow \varphi \in \ell_\mu(s,j)^i \Rightarrow \varphi \in \ell_\tau(r,j).$$

  In the latter case, by K4, we must have $L_i\neg\alpha \in \ell_\mu(s,j)$, and by the definition of $\rho^i_\mu$, there must be some $t \in R_\tau$ such that $\rho^i_\mu(t) = (s,j)$ and $\alpha \notin \ell_\tau(t,j)$, giving us a contradiction. Consequently we must have $\varphi \in \ell_\tau(r,j)$.

For the converse, suppose $\varphi \in \ell_\tau(r,j)$. Again we consider two possibilities:

- For all $t \neq r$ such that $\pi_r(j)_i = \pi_t(j)_i$, $t \in R_{\tau\backslash i}$ and $\rho^i_\tau(t) = (r,j)$. In this case by the definition of $\rho^i_\tau$ and $\prec_i$, we have $\ell_\tau(r,j)^i \subseteq \ell_{\tau\backslash i}(t,j)^i$. Consequently $\alpha \in \ell_{\tau\backslash i}(t,j)^i$, and by K3, $\alpha \in \ell_\tau(r,j)$. By the inductive hypothesis, for all $t$ such that $\pi_r(j)_i = \pi_t(j)_i$, we have $M_F, \pi_t, j \models \alpha$, so $M_F, \pi_r, j \models K_i\alpha$.
- There is some $\mu$ and some $s \in T_\mu$ such that $\mu\backslash i = \tau$ and for all $t$, such that $\pi_r(j)_i = \pi_t(j)_i$, we have $\rho^i_\mu(t) = (s,j)$. For all such $t$ we have $\ell_\tau(t,j) \prec_i \ell_\mu(s,j)$, so by the definition of $\prec_i$, $K_i\alpha \in \ell_\mu(s,j)$ implies $\alpha \in \ell_\tau(t,j)$. Since $K_i\alpha \in \ell_\tau(r,j)$ implies $K_i\alpha \in \ell_\mu(s,j)$, it must be that for all $t$, such that $\pi_r(j)_i = \pi_t(j)_i$, $\alpha \in \ell_\tau(t,j)$ and by the inductive hypothesis $M_F, \pi_t, j \models \alpha$. Thus $M_F, \pi_r, j \models K_i\alpha$.

▮

# 8 Conclusion

In this paper we have present sound and complete axiomatizations for logics of knowledge and past time with the synchronization and unique initial states constraints. We note here that the axiomatization for Synchronization and UIS is a straightforward combination of the two, and that the proof of completeness can be easily modified to accommodate this.

For future work we will be investigating the semantic restrictions of *perfect recall* (where an agent retains the knowledge of previous times), and *no learning* (where an agent's knowledge can not increase over time) [1]. We will also look at incorporating common knowledge into the language, and extending the axiomatizations to combinations of these semantic restrictions.

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