

Bisimulation Quantified Logics: Undecidability

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Abstract. In this paper we introduce a general semantic interpretation for propositional quantification in all multi-modal logics based on bisimulations (bisimulation quantification). Bisimulation quantification has previously been considered in the context of isolated modal logics, such as PDL (D’Agostino and Hollenberg, 2000), intuitionistic logic (Pitts, 1992) and logics of knowledge (French 2003). We investigate the properties of bisimulation quantifiers in general modal logics, particularly the expressivity and decidability, and seek to motivate the use of bisimulation quantified modal logics. This paper addresses two important questions: when are bisimulation quantified logics bisimulation invariant; and do bisimulation quantifiers always preserve decidability? We provide a sufficient condition for bisimulation invariance, and give two examples of decidable modal logics which are undecidable when augmented with bisimulation quantifiers. This is part of a program of study to characterize the expressivity and decidability of bisimulation quantified modal logics.

1 Introduction

In this paper we introduce a general semantic interpretation for propositional quantification in modal logic. This interpretation is based on the notion of *bisimulation* [11, 10]. We use *bisimulation quantifiers* [3] to quantify over the interpretation of propositional atoms in all bisimilar models. Bisimulation quantifiers were introduced in [8] and [14] and have been defined in logics based on PDL [3], intuitionistic logics [12] and logics of knowledge [6].

Modal logics find use in great variety of applications, such as temporal reasoning, reasoning about the correctness of programs, and reasoning about knowledge [4]. The variety of modal logics is achieved by restricting the structures (or models) of the logics to various classes. When applying bisimulation quantifiers to these logics we do not quantify over the entire bisimulation class of a structure, rather we quantify over the intersection of that bisimulation class with the class of structures that defines the logic.

In the context of modal logic, bisimulation quantifiers are a natural extension which have some nice properties which we discuss in Section 3. We would like bisimulation quantification to preserve our intuitions regarding propositional quantification, particularly the axioms of *existential introduction* and *existential elimination*. We define a class of logics, the *safe* logics, for which these axioms are sound.

In 1970 Kit Fine [5] investigated the decidability of propositional quantifiers in modal logics. The standard propositional quantifiers were highly expressive and often undecidable (for example, in the cases of **K** and **S4**). Bisimulation quantifiers quantify over the interpretation of propositions in all bisimilar models and are consequently

less expressive. In fact the logic **K** augmented with bisimulation quantifiers is no more expressive than **K** itself.

D'Agostino and Hollenberg have shown the decidability of **BQL** [3], which is effectively the dynamic modal logic, **PDL**, augmented with bisimulation quantifiers, and in [6] the decidability of logics of knowledge with bisimulation quantifiers is shown. Unfortunately it is not always the case that augmentation with bisimulation quantifiers preserves decidability. In this paper we show two decidable modal logics: **S5** \times **S5** and **LTL** \times **S5**, are undecidable when augmented with bisimulation quantifiers.

2 Syntax and Semantics

We let L_C be a multi-modal logic consisting of k modalities, where C represents the class of frames over which the logic is defined. Given a modal logic L_C , we will let QL_C be an extension of L_C including *bisimulation quantifiers* (defined below).

Let \mathcal{V} be a set of atomic propositions. We recursively define the formulas of L_C as follows:

$$\alpha ::= x \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \diamond_i \alpha \quad (1)$$

where $x \in \mathcal{V}$ and $i = 1, \dots, k$. The syntax for QL_C includes the recursion $\exists x\alpha$, where $x \in \mathcal{V}$. We let the abbreviations \wedge , \rightarrow , \leftrightarrow , \top , \perp and \forall be defined as usual and let $\square_i \alpha$ abbreviate $\neg \diamond_i \neg \alpha$.

We will first give the semantics for an arbitrary modal logic without bisimulation quantifiers.

Definition 1. A k -frame, F , is given by the tuple (S, R_1, \dots, R_k) where S is a set of worlds and for each i , $R_i \subseteq S \times S$. A k -model, M , is given by the tuple $(S, R_1, \dots, R_k, \pi, s)$ where $s \in S$ and $\pi : S \rightarrow \wp(\mathcal{V})$

We let \bar{R} abbreviate R_1, \dots, R_k , and for $s' \in S$ we let $M_{s'} = (S, \bar{R}, \pi, s')$. Given a k -model, $M = (S, \bar{R}, \pi, s)$, the semantic interpretation of propositional atoms and modalities is given by:

$$M \models x \iff x \in \pi(s) \quad (2)$$

$$M \models \diamond_i \alpha \iff \exists (s, t) \in R_i, M_t \models \alpha \quad (3)$$

and the propositional operators have their usual meaning. If a formula is true in every k -model it is referred to as a validity, and if it is true in some k -model it is satisfiable. The set of valid formulas in this language is referred to as K_k (the fusion of k unrestricted modalities, see [7]). However the usefulness of modal logic comes from placing restrictions on modalities. For example specifying a modality to be transitive, irreflexive and antisymmetric allows it to represent properties of time, and specifying a modality to be reflexive, symmetric and transitive allows it to represent properties of knowledge [4].

For a given k , we let \mathcal{C} be a class of k -frames.

Definition 2. Given a set of k -frames, \mathcal{C} , we say (S, \bar{R}) is a \mathcal{C} -frame if $(S, \bar{R}) \in \mathcal{C}$ and we define a \mathcal{C} -model to be the tuple $M = (S, \bar{R}, \pi, s)$ where $\pi : S \rightarrow \wp(\mathcal{V})$, $s \in S$ and $(S, \bar{R}) \in \mathcal{C}$.

The set of valid formulas in the language $L_{\mathcal{C}}$ is defined by restricting the logic to the class of \mathcal{C} -frames, so that a formula is valid for $L_{\mathcal{C}}$ if and only if it is true for all \mathcal{C} -frames. To define propositional quantification we will require some additional definitions, based on the concept of a bisimulation [10] [11].

Definition 3. Given the \mathcal{C} -models $M = (S, \overline{R}, \pi, s)$ and $N = (T, \overline{P}, \lambda, t)$, and given $\Theta \subseteq \mathcal{V}$ we say the models M and N are Θ -bisimilar (written $M \cong_{\Theta} N$) if there is some relation $B \subseteq S \times T$ such that:

1. $(s, t) \in B$ and for all $(s, t) \in B$, $\pi(s) \setminus \Theta = \lambda(t) \setminus \Theta$;
2. for all $(s, t) \in B$, for all $u \in S$, if $(s, u) \in R_i$ then there exists some $(u, v) \in B$ such that $(t, v) \in P_i$;
3. for all $(s, t) \in B$, for all $v \in T$ if $(t, v) \in P_i$ then there exists some $(u, v) \in B$ such that $(s, u) \in R_i$.

We call such a relation B a Θ -bisimulation¹ from M to N . If $M \cong_{\emptyset} N$ we say they are bisimilar (written $M \cong N$), and if $M \cong_{\{x\}} N$ we say M and N are x -bisimilar (written $M \cong_x N$).

We are now able to give the semantic interpretation of bisimulation quantification in $QL_{\mathcal{C}}$:

$M \models \exists x \alpha$ if and only if there is some \mathcal{C} -model, N such that $M \cong_x N$ and $N \models \alpha$.

We note that the meaning of \models is now dependent on \mathcal{C} . In the case that \mathcal{C} is not clear from context, we will write $\models_{\mathcal{C}}$.

3 Properties of bisimulation quantification

Bisimulations have been investigated in the context of modal logics for many years. The following results are well-known:

Lemma 1. For all $\Theta \subseteq \mathcal{V}$, Θ -bisimulation is an equivalence relation.

Lemma 2. For all pure modal formulas ϕ not containing atoms from Θ , for all models, M and N , if $M \models \phi$ and M is Θ -bisimilar to N , then $N \models \phi$.

Bisimulation quantifiers are a natural extension to modal logic. They allow us to achieve some powers of monadic second-order logic whilst retaining many of the intuitions of pure modal logic. The semantics can appear daunting, since every occurrence of a quantifier in a formula requires us to consider all possible bisimulations of a given model (this complexity is apparent in Section 4.1). However there is a good argument for studying bisimulation quantifiers further. Several bisimulation quantified modal logics, such as BQL [3], are expressively equivalent to the modal μ -calculus. Reasoning

¹ Note, in some previous work (e.g. [3]), a Θ -bisimulation refers to what we would denote a $\mathcal{V} \setminus \Theta$ -bisimulation. The current notation is more convenient in the context of propositional quantification.

in the μ -calculus is relatively efficient (EXPTIME), and μ -automata [9] allow us to effectively represent bisimulation quantifiers (the construction is effectively equivalent to the projection operation in binary tree automata). In such a case we can avoid much of the complexity involved in bisimulations, whilst still enjoying the ability to express higher order properties.

The action of bisimulation quantifiers is also worth investigating in its own right, rather than as a simple tool to gain greater expressivity. Given any structure, M , in any pure modal logic, \mathcal{C} , let \mathcal{L} be set of formulas ϕ such that $M \models \phi$. We will refer to the set \mathcal{L} as the *facts* of M , whilst \mathcal{C} is the context of M . We can suppose that an agent reasoning about M knows the context of M (for example if \mathcal{C} was a temporal system, we would expect an agent would know time is transitive), and knows all the facts of M . Importantly the agent does not know about the structure of M , which is really just a tool to facilitate the set \mathcal{L} . So if the agent were to reason about alternative interpretations for an atom, x , we would expect the agent to consider any model that is firstly, an element of \mathcal{C} , and secondly agrees with \mathcal{L} on all pure modal formulas not containing x . This process can be applied recursively to motivate any number of nestings of bisimulation quantifiers. This argument is not precise: there are cases where two non-bisimilar models can satisfy the same set of pure modal formulas. However it does give some philosophical motivation for studying bisimulation quantifiers. The relationship between bisimulations and non-well-founded sets is explored in [2].

The final reason for examining bisimulation quantifiers is that they give us some power for describing the logic itself. For example, $\forall x(x \rightarrow \diamond_i x)$ is equivalent to saying “the modality, \diamond_i , is reflexive”. However this is not a statement about any particular structure. It says that in every model in the class \mathcal{C} the modality \diamond_i is reflexive, so $x \rightarrow \diamond_i x$ is a validity. Being able to express validities as validities, rather than simply as satisfied formulas, does not change the expressivity of the logic, but it certainly could be significant in providing axiomatizations, or allowing meta-logical reasoning.

As the properties definable in modal logic are bisimulation invariant [13], the application of bisimulation quantifiers does not affect the interpretation of pure modal formulas. However we would also like the semantic interpretation of bisimulation quantifiers to preserve the intuitions of propositional quantification. Particularly, it should satisfy the standard axioms for propositional quantifiers:

1. If $\phi \rightarrow \psi$ is a validity and ψ does not contain the variable x , then for every model that $\exists x\phi \rightarrow \psi$ should also be a validity. This is referred to as *existential elimination*.
2. Suppose α is a formula such that β is free for x in α . Then $\alpha[x\backslash\beta] \rightarrow \exists x\alpha$ is a validity. This is referred to as *existential introduction*.

Here $\alpha[x\backslash\beta]$ is the formula α with every free occurrence of the variable x replaced by the formula β , and β is *free for x in α* if and only if for every free variable, y , of β the variable x is not in the scope of a quantifier, $\exists y$, in α .

Unfortunately these axioms will not hold for all logics, $QL_{\mathcal{C}}$. However these axioms are sound for all *safe* logics, defined below ²:

² The author thanks Giovanna D’Agostino and Giacomo Lenzi for improving this definition

Definition 4. We say the class of frames \mathcal{C} is safe if and only if for any $\Theta_1, \Theta_2 \subset \mathcal{V}$, for any \mathcal{C} -models M and N such that $M \cong_{\Theta_1 \cup \Theta_2} N$, there is some \mathcal{C} -model, K such that $M \cong_{\Theta_1} K$ and $N \cong_{\Theta_2} K$.

Lemma 3. Suppose that \mathcal{C} is safe and M, N are \mathcal{C} -models such that $M \cong_{\Theta} N$. Then for all formulas, α , not containing free atoms from Θ , $M \models \alpha$ if and only if $N \models \alpha$.

Proof. This is shown by induction over the complexity of formulas. The cases for propositional atoms and propositional operators are trivial. Suppose that for some α not containing atoms from Θ , for all \mathcal{C} -models, M and N with $M \cong_{\Theta} N$ we have $M \models \alpha$ if and only if $N \models \alpha$.

Let $M = (S, \overline{R}, \pi, s)$ and $N = (T, \overline{P}, \mu, t)$ be \mathcal{C} -models. If $M \cong_{\Theta} N$ and $M \models \diamond_i \alpha$, then there is some $s' \in S$ such that $(s, s') \in R_i$ and $(S, \overline{R}, \pi, s') \models \alpha$. Since $M \cong_{\Theta} N$, there is some $t' \in T$ such that $(t, t') \in P_i$ and $(S, \overline{R}, \pi, s') \cong_{\Theta} (T, \overline{P}, \mu, t')$. By the induction hypothesis, $(T, \overline{P}, \mu, t') \models \alpha$, and thus $N \models \diamond_i \alpha$.

Now suppose that M and N are \mathcal{C} -models such that $M \cong_{\Theta} N$ and $M \models \exists x \alpha$. Then there is some model M' such that $M \cong_x M'$ and $M' \models \alpha$. Therefore N and M' are $\{x\} \cup \Theta$ -bisimilar, (by Lemma 1). Since \mathcal{C} is safe, there must be a \mathcal{C} -frame, K such that $K \cong_{\Theta} M'$ and $K \cong_{\{x\}} N$. By the induction hypothesis, $K \models \alpha$, and thus $N \models \exists x \alpha$. As the converse for these inductions is symmetric this is sufficient to prove the lemma.

Lemma 4. Given \mathcal{C} is safe, the axioms existential elimination and existential introduction are sound for $\text{QL}_{\mathcal{C}}$.

Proof. To show existential elimination is sound, suppose that for all \mathcal{C} -models, $\alpha \rightarrow \beta$ is a validity, and for some \mathcal{C} -model, M , we have $M \models \exists x \alpha$, where x is not a variable of β . Thus there is some \mathcal{C} -model, N , such that $M \cong_x N$ and N models α . Since $\alpha \rightarrow \beta$ is a validity, we have $N \models \beta$. Since $M \cong_x N$, it follows from Lemma 3 that $M \models \beta$. Therefore existential elimination is sound.

To show existential introduction suppose that $M = (S, \overline{R}, \pi, s)$ is a \mathcal{C} -model such that $M \models \alpha[x \setminus \beta]$ where β is free for x in α . We define the model $N = M_{\beta \rightarrow x} = (S, \overline{R}, \rho, s)$ where ρ is such that for all $t \in S$, $\pi(t) \setminus \{x\} = \rho(t) \setminus \{x\}$ and $x \in \rho(t)$ if and only if $M_t \models \beta$. In the first instance we will assume that x is not a free variable of β . For all subformulas, γ of α , let $\gamma' = \gamma[x \setminus \beta]$. We show for all $t \in S$, $M_t \models \gamma'$ if and only if $N_t \models \gamma$ by induction over the complexity of formulas. As the base cases of the induction we have $M_t \models \gamma'$ if and only if $N_t \models \gamma$ where γ is an atomic proposition. Now assume for any γ we have $M_t \models \gamma'$ if and only if $N_t \models \gamma$. It follows directly that

1. $N_t \models \neg \gamma \iff M_t \models (\neg \gamma)'$.
2. $N_t \models \gamma_1 \vee \gamma_2 \iff M_t \models (\gamma_1 \vee \gamma_2)'$.
3. $N_t \models \diamond_i \gamma \iff M_t \models (\diamond_i \gamma)'$.

Now suppose that $N_t \models \exists y \gamma$. Therefore there is some \mathcal{C} -model $K = (U, \overline{P}, \rho, u)$ such that $K \cong_{\{y\}} N_t$ (so $K \cong_{\{x, y\}} M_t$) and $K \models \gamma$. By the safety of \mathcal{C} there is some \mathcal{C} -model, L such that $L \cong_x K$ and $L \cong_y M_t$. If x does not occur free in γ , then by Lemma 3 we have $L \models \gamma$, and $\gamma = \gamma'$, so the induction follows. If x does occur in γ , then β does not contain the atom y , since x is free for β in α . Since x does not occur in β , we have for all $v \in U$, $K_v \models x \leftrightarrow \beta$. Therefore $K = K_{\beta \rightarrow x}$ so we can apply this

construction inductively to derive $K \models \gamma'$. It follows from Lemma 3 that $L \models \gamma'$ and therefore $M_t \models \exists y \gamma'$.

Conversely suppose that $M_t \models \exists y \gamma'$. There is some \mathcal{C} -model, $K = (U, \overline{P}, \rho, u)$ such that $K \cong_{\{y\}} M_t$ (so $K \cong_{\{x,y\}} N_t$) and $K \models \gamma'$. By the safety of \mathcal{C} there is some model $L = (V, \overline{Q}, \eta, v)$ such that $L \cong_{\{y\}} N_t$ and $L \cong_{\{x\}} K$. If γ does not contain x the result follows from Lemma 3. If γ does contain x , then β cannot contain y , so for all $w \in V$, $L_w \models \beta \leftrightarrow x$ (since L is y -bisimilar to N_t). As $K \models \gamma'$, it follows from Lemma 3 that $L \models \gamma'$. Since $L = L_{\beta \rightarrow x}$ we can again apply this construction inductively to derive $L \models \gamma$ and therefore $N_t \models \exists y \gamma$.

To complete this proof we need to generalize to the case where x may be a variable of β . Let α' be the formula with every free occurrence of x replaced by y , where y does not occur in α nor β . As x is free for β in α , y is free for β in α' (since y does not occur in α). Clearly $\alpha'[y \setminus \beta]$ is the same as $\alpha[x \setminus \beta]$ and as y does not occur in β the above induction applies. Thus

$$\alpha[x \setminus \beta] \longrightarrow \exists y \alpha' \quad (4)$$

is a validity. As x does not occur free in α' from the semantic definition of existential quantification and Lemma 3 we have $\exists y \alpha' \rightarrow \exists x \alpha$ is also a validity and hence existential introduction is valid.

To see the value of the lemma above it is worthwhile looking and a class of frames which is not safe. Later we will see a logic $\mathbf{S5} \times \mathbf{S5}$ which does not enjoy this property, but first we will consider a simpler logic. Let **Three** be the set of all frames, $F = (S, R)$ where $|S| = 3$ and $R = S \times S$. We can define two **Three**-models which demonstrate existential elimination is not sound. Particularly, let

- $M = (S, R, \pi, a)$ where $S = \{a, b, c\}$, $R = S \times S$, $\pi(a) = \{y, z\}$, $\pi(b) = \{y\}$ and $\pi(c) = \{x\}$.
- $N = (T, P, \rho, d)$ where $T = \{d, e, f\}$, $P = T \times T$, $\rho(d) = \{y\}$, $\rho(e) = \{x, w\}$ and $\rho(f) = \{x\}$.

Now $M \cong_{\{w,z\}} N$ via the bisimulation, $B = \{(a, d), (b, d), (c, e), (c, f)\}$ but there is no **Three**-model, $K = (U, Q, \eta, u)$ such that $M \cong_{\{w\}} K$ and $N \cong_{\{z\}} K$. To see this we note that any such model must contain some state $g \in U$, with $\eta(g) = \{y, z\}$, some state $h \in U$ with $\eta(h) = \{y\}$, some state $i \in U$ with $\eta(i) = \{x, w\}$ and some state $j \in U$ with $\eta(j) = \{x\}$. That is, K cannot be a **Three**-model. Therefore $M \models_{\mathbf{Three}} \exists z \exists w (\diamond(x \wedge w) \wedge \diamond(x \wedge \neg w))$, but $M \not\models_{\mathbf{Three}} \exists w (\diamond(x \wedge w) \wedge \diamond(x \wedge \neg w))$

This example exploits a simple counting property to invalidate existential elimination. The $\mathbf{S5} \times \mathbf{S5}$ example below shows how more complex structural properties can make a class of frames unsafe.

4 Undecidable logics

Here we will briefly examine some logics which are decidable, but whose bisimulation quantified extension is undecidable. The logics we will describe are $\text{PLTL} \times \mathbf{S5}$ and $\mathbf{S5} \times \mathbf{S5}$. The decidability of both of these logics is described in [7]. Both undecidability

proofs will make use of tiling problems, which is a common technique for proving the undecidability of modal logics.

The tiling problem is as follows: We are given a finite set $\Gamma = \{\gamma_i \mid i = 1, \dots, m\}$ of tiles. Each tile γ_i has four coloured sides: left, right, top and bottom, written γ_i^l , γ_i^r , γ_i^t , and γ_i^b . Each side can be one of n colours c_j for $j = 1, \dots, n$. Given any set of these tiles, we would like to know if we can cover the plane $\mathbb{N} \times \mathbb{N}$ with these tiles such that adjacent sides share the same colour. Formally, given some finite set of tiles Γ we would like to decide if there exists a function $\lambda : \mathbb{N} \times \mathbb{N} \rightarrow \Gamma$ such that for all $(x, y) \in \mathbb{N} \times \mathbb{N}$

1. $\lambda(x, y)^r = \lambda(x + 1, y)^l$
2. $\lambda(x, y)^t = \lambda(x, y + 1)^b$

where $\lambda(x, y)^t$ is the colour of the top side of the tile on (x, y) , and likewise for the other sides. As shown by Berger [1], this problem is undecidable.

4.1 $\mathbf{S5} \times \mathbf{S5}$

The logic $\mathbf{S5} \times \mathbf{S5}$ is defined to be the cross-product of two $\mathbf{S5}$ frames. The syntax is given by $\alpha ::= x \mid \alpha \vee \alpha \mid \neg \alpha \mid \diamond_1 \alpha \mid \diamond_2 \alpha$. The logic is defined over the set of all frames specified as follows: $F = (S, R_1, R_2)$ where

- $S = S_1 \times S_2$ where S_1 and S_2 are arbitrary non-empty sets.
- $((a, b), (c, d)) \in R_1$ if and only if $a = c$.
- $((a, b), (c, d)) \in R_2$ if and only if $b = d$.

By an abuse of notation, we will also refer to the class of $\mathbf{S5} \times \mathbf{S5}$ frames as $\mathbf{S5} \times \mathbf{S5}$.

Lemma 5. *The class of frames, $\mathbf{S5} \times \mathbf{S5}$, is not safe.*

Proof. We will prove this lemma with an example. In Figure 4.1 there are two $\mathbf{S5} \times \mathbf{S5}$ -

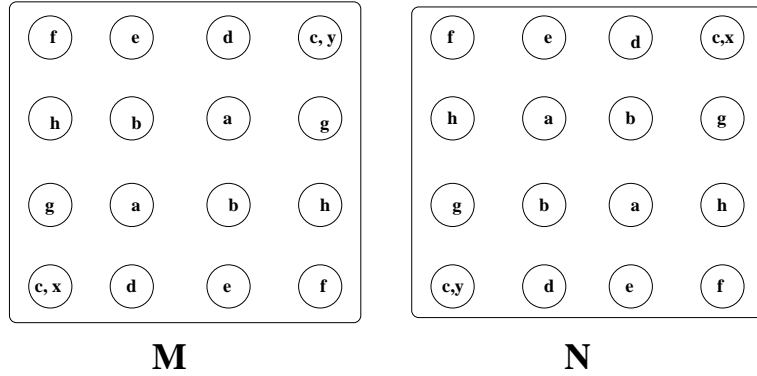


Fig. 1. An example where $\mathbf{S5} \times \mathbf{S5}$ is not safe. The two modalities correspond to the horizontal and vertical axis respectively.

models, M and N . One modality corresponds to the vertical axis, and one modality corresponds to the horizontal access. The propositions true at each state are marked, and we let the starting state for each model be the bottom left state. We can see the two models $\{x, y\}$ -bisimilar, via a bisimulation which relates states with the same propositions (excepting x and y). Note that when we ignore x and y , the two models are almost identical, except the central four states are transposed.

Now suppose for contradiction that there is some $\mathbf{S5} \times \mathbf{S5}$ -model K such that $K \cong_{\{y\}} M$ and $K \cong_{\{x\}} N$. Therefore the starting state for K must be labeled with the propositions, c , x and y . Since K is y bisimilar to M , by Lemma 2 for every pure modal formula, ϕ , not containing y K satisfies ϕ if and only if M satisfies ϕ (and likewise for N and x). Let

$$\phi(w, z) = \Box_1(d \rightarrow \Box_2(w \rightarrow \Box_1(g \rightarrow \Box_2(c \rightarrow \neg z))).$$

We can see $M \models \phi(b, x)$ and $N \models \phi(a, y)$, so $K \models \phi(b, x) \wedge \phi(a, y)$. However since K is an $\mathbf{S5} \times \mathbf{S5}$ -model, $\{x, y\}$ -bisimilar to both M and N , there must be states h, i, j, k in K where:

1. h is the starting state, labeled with c, x, y ;
2. i is some state such that $(h, i) \in R_1$ and i is labeled with d ;
3. j is some state such that $(h, j) \in R_2$ and j is labeled with g ;
4. k is defined such that $(i, k) \in R_2$ and $(j, k) \in R_1$.

By observing M and N we can see that the state k must be labeled with either a or b . However the relations R_1 and R_2 are symmetric, so if k was labeled by a we would have $K \models \neg\phi(a, y)$, and if k was labeled by b we would have $K \models \neg\phi(b, x)$, giving the necessary contradiction.

As $\mathbf{S5} \times \mathbf{S5}$ is not safe we must proceed with caution as our usual intuitions regarding the behavior of quantification will not necessarily hold. However we can see from the proof above that bisimulation invariance still applies to pure modal formulas.

We encode the tiling problem by defining propositional atoms u and v such that they allow us to linearly order the horizontal and vertical axis. In particular we would like the following properties to hold:

1. if u is true at any state (a, b) in the model, then there is exactly one state (a, c) in the model such that $c \neq b$ where v is true.
2. if v is true at any state (a, b) in the model, then there is exactly one state (c, b) in the model such that $c \neq a$ where u is true.

We will refer to such properties as the *step* properties. Such a configuration is given in Figure 4.1.

In general bisimulation quantifiers do not allow us to define such strict properties, but we will see that we can “simulate” such properties in the scope of quantifiers.

For each $\gamma \in \Gamma$ we suppose that there is a unique propositional atom in \mathcal{V} , which we also refer to as γ (where its meaning shall be clear from context). We encode the tiling problem in several stages: Let

$$\text{step}(x, y) = x \wedge \Box_2 \neg y \wedge \Box_1 \Box_2 ((x \rightarrow \Diamond_1 y) \wedge (y \rightarrow \Diamond_2 x)).$$

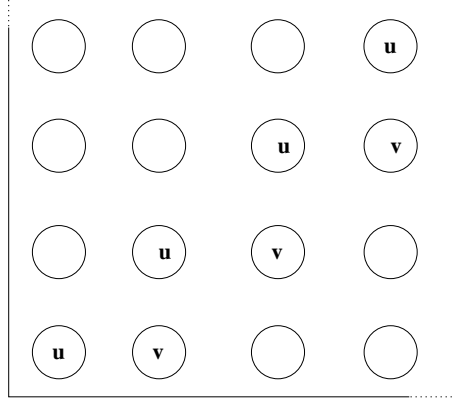


Fig. 2. u and v define the step property in $\mathbf{S5} \times \mathbf{S5}$, which allows us to discretize and order the horizontal and vertical axis.

This states that x and y satisfy the step properties except for the uniqueness constraints. Next we define

$$step(x, y, u, v) = u \wedge \Box_2 \neg v \wedge \Box_1 \Box_2 ((u \rightarrow (x \wedge \Diamond_1 v)) \wedge (v \rightarrow (y \wedge \Diamond_2 u))).$$

Finally assuming the u and v satisfy the step properties we can encode the tiling problem as follows:

$$\begin{aligned} right &= \Box_1 \Box_2 \left(\bigwedge_{\gamma \in \Gamma} (\gamma \rightarrow \forall z (z \rightarrow \Box_2 (u \rightarrow \Box_1 (v \rightarrow \Diamond_2 (\Diamond_1 z \wedge \bigvee_{\gamma^r = \delta^l} \delta)))))) \right) \\ up &= \Box_1 \Box_2 \left(\bigwedge_{\gamma \in \Gamma} (\gamma \rightarrow \forall z (z \rightarrow \Box_1 (v \rightarrow \Box_2 (u \rightarrow \Diamond_1 (\Diamond_2 z \wedge \bigvee_{\gamma^t = \delta_b} \delta)))))) \right) \\ unique &= \Box_1 \Box_2 \left(\left(\bigvee_{\gamma \in \Gamma} \gamma \right) \wedge \left(\bigwedge_{\gamma \in \Gamma} \gamma \rightarrow \bigwedge_{\delta \in \Gamma - \gamma} \neg \delta \right) \right) \end{aligned}$$

We now give the complete formula as

$$Tile_{\Gamma} = unique \wedge step(x, y) \wedge \forall u \forall v (step(x, y, u, v) \rightarrow (right \wedge up)).$$

We note that the alternation in quantifier is similar to a least fixed point. If $Tile_{\Gamma}$ is true then there must be some uv -bisimulation which satisfies the step properties, and if $Tile_{\Gamma}$ is false, then it must be false for some xy -bisimulation where x and y satisfy the step properties (and hence u and v satisfy the step properties up to bisimulation equivalence).

Lemma 6. $Tile_{\Gamma}$ is satisfiable if and only if Γ can tile the plane.

Proof. First, let us suppose that there is a tiling, λ , of the plane with the tiles in Γ . We show that there is a model, $M = (S, R_1, R_2, \pi, s)$, satisfying $Tile_\Gamma$. We let:

1. $S = \omega \times \omega$
2. $R_1 = \{((a, b), (c, b)) \mid a, b, c \in \omega\}$
3. $R_2 = \{((a, b), (a, c)) \mid a, b, c \in \omega\}$
4. $\gamma \in \pi(a, b) \Leftrightarrow \gamma = \lambda(a, b)$, $x \in \pi(a, b) \Leftrightarrow a = b$ and $y \in \pi(a, b) \Leftrightarrow a = b + 1$.
5. $s = (0, 0)$.

As λ is a function it follows that $M \models \text{unique}$, and by the construction of M we also have $M \models \text{step}(x, y)$. The remaining part of the formula, $Tile_\Gamma$ is in the scope of universal quantifiers which makes things more complicated. It is especially complicated in the case of $\mathbf{S5} \times \mathbf{S5}$ as the logic is not safe. However we have constructed a model where every state can be uniquely identified by a formula, and we will use these “unique” formulas to show that $Tile_\Gamma$ is satisfied. The unique formulas are defined recursively by

$$\begin{aligned}\eta_0 &= x \wedge \Box_2 \neg y \\ \eta_{i+1} &= x \wedge \Diamond_2 (y \wedge \Diamond_1 \eta_i) \\ \eta(a, b) &= \Diamond_2 \eta_a \wedge \Diamond_1 \eta_b.\end{aligned}$$

We can see $M_t \models \eta_a$ if and only if $t = (a, a)$, and thus for all $t \in S$, $M_t \models \eta(a, b)$ if and only if $t = (a, b)$. Consequently, for any $(a, b) \in S$, we have $M \models \eta(a, b) \rightarrow \lambda(a, b)$. By a result of van Benthem, pure modal formulas are bisimulation invariant, so for any model N where $N \cong_{u, v} M$ we will have $N \models \Box_1 \Box_2 (\eta((a, b) \rightarrow \lambda(a, b)))$. Let $N = (S', R'_1, R'_2, \pi', s')$ be any $\{u, v\}$ -bisimulation of M , such that N is an $\mathbf{S5} \times \mathbf{S5}$ -model and $N \models \text{step}(x, y, u, v)$. For any state $t \in S'$, there is a unique (a, b) such that $N_t \models \eta(a, b)$ and thus $N_t \models \lambda(a, b)$. Suppose that $\lambda(a, b) = \gamma$. To show that the subformula *right* is satisfied by N we must show

$$N_t \models \forall z (z \rightarrow \Box_1 (v \rightarrow \Box_2 (u \rightarrow \Diamond_1 (\Diamond_2 z \wedge \bigvee_{\gamma^t = \delta_b} \delta)))) \quad (5)$$

For any z -bisimulation $N' = (T, P_1, P_2, \rho, t')$ of N_t where $N' \models z$, we will have $N' \models \Box_2 (u \rightarrow \eta(a, a))$ (since $u \rightarrow x$). Since $N \models \text{step}(x, y, u, v)$, there is some state $e \in T$ such that $N'_e \models u$ where $(t', e) \in P_2$. By the definition of the step function we also have $N' \models \Box_2 (u \rightarrow \Box_1 (v \rightarrow \eta(a + 1, a)))$, (since $u \rightarrow x$ and $v \rightarrow y$ in N'). Let $f \in T$ be any state such that $N'_f \models v$ and $(e, f) \in P_1$. Since N is a $\mathbf{S5} \times \mathbf{S5}$ frame there is some unique state $g \in T$ such that $(f, g) \in P_2$ and $(t', g) \in P_1$. Also since $N'_f \models \Diamond_2 \eta_{a+1}$ and $N'_e \models \Diamond_1 \eta_b$, we have $N'_g \models \eta(a + 1, b)$ and hence $N'_g \models \lambda(a + 1, b) \wedge \Diamond_1 z$. Therefore we have shown (5) to be true, and as t was chosen arbitrarily it follows that $N \models \text{right}$. The case for showing that $N \models \text{up}$ is symmetric. As N is an arbitrary $\{u, v\}$ -bisimulation of M satisfying $\text{step}(x, y, u, v)$ it follows that $M \models \forall u \forall v (\text{step}(x, y, u, v) \rightarrow (\text{right} \wedge \text{up}))$.

Conversely, suppose that $M \models Tile_\Gamma$, where $M = (S, R_1, R_2, \pi, s)$. Since $M \models Tile_\Gamma$ for every $\{u, v\}$ -bisimulation, N of M we have $N \models \text{unique} \wedge \text{step}(x, y) \wedge \text{step}(x, y, u, v) \rightarrow (\text{right} \wedge \text{up})$. Clearly there is some such $\{u, v\}$ -bisimulation $N =$

(T, P_1, P_2, ρ, t) of M such that $N \models \text{step}(x, y, u, v)$. We define a function $\phi : \omega \times \omega \rightarrow T$ inductively by

1. $\phi(0, 0) = t$. We note that as $N \models \text{step}(x, y, u, v)$ we have $N_{\phi(0,0)} \models u \wedge \diamond_1 v$.
2. Given $a \in \omega$ such that $\phi(a, a) = t'$ and $N_{\phi(a,a)} \models u \wedge \diamond_1 v$ we choose some $r \in T$ such that $u \in \rho(r)$ and for some $r' \in T, v \in \rho(r'), (t', r') \in P_1$ and $(r', r) \in P_2$. We let $\phi(a+1, a+1) = r$, and note that $N_{\phi(a+1,a+1)} \models u \wedge \diamond_1 v$ (since $N \models \text{step}(x, y, u, v)$). Thus for all $a \in \omega$ we can define $\phi(a, a)$.
3. For any $(a, b) \in \omega \times \omega$ where $a \neq b$, we define $\phi(a, b)$ to be the unique element r of T , such that $(\phi(a, a), r) \in P_1$ and $(\phi(b, b), r) \in P_2$

It follows from the definitions of *up* and *right*, that λ is a tiling of the plane where $\lambda(a, b) = \gamma$ if and only if $\gamma \in \rho(\phi(a, b))$.

Corollary 1. $\text{QL}_{\text{S5} \times \text{S5}}$, the bisimulation quantified extension of $\text{S5} \times \text{S5}$, is undecidable.

4.2 LTL \times S5

In the previous section we saw a complicated bisimulation quantified logic which was neither safe nor decidable. We might hope that bisimulation quantifiers preserve the decidability of all safe logics, but this is not the case. $\text{LTL} \times \text{S5}$ is a safe logic which is also undecidable when augmented with bisimulation quantifiers.

The syntax for $\text{LTL} \times \text{S5}$ is as follows:

$$\alpha ::= x|\alpha \vee \alpha|\neg\alpha|X\alpha|F\alpha|\Box\alpha$$

where $G\alpha$ is the dual of $F\alpha$.

The logic $\text{LTL} \times \text{S5}$ is defined over structures specified by $F \subset \mathcal{F} = \{\sigma | \sigma : \mathbb{N} \rightarrow \wp(\mathcal{V})\}$. The states of F are represented by the tuple, σ, i , where $\sigma \in F$ and $i \in \mathbb{N}$. The semantics are:

$$\begin{aligned} (F, \sigma, j) &\models x \Leftrightarrow x \in \sigma(j), \text{ for all } x \in \mathcal{V}. \\ (F, \sigma, j) &\models X\alpha \Leftrightarrow (F, \sigma, j+1) \models \alpha. \\ (F, \sigma, j) &\models G\alpha \Leftrightarrow \forall k \geq j, (F, \sigma, k) \models \alpha. \\ (F, \sigma, j) &\models \Box\alpha \Leftrightarrow \forall \sigma', (F, \sigma', j) \models \alpha \end{aligned}$$

where \vee and \neg have their usual meaning.

This logic has been shown to be decidable (see [7]). We note that the set $\text{LTL} \times \text{S5}$ models can easily be translated to define a class of frames in the notation of Section 2. This allows us to provide the semantics for bisimulation quantifiers. We give the following lemmas.

Lemma 7. $\text{LTL} \times \text{S5}$ is safe.

Proof. Suppose that $M = (F, \sigma, 0)$ and $N = (G, \eta, 0)$ are $\Theta \cup \Lambda$ -bisimilar models, via the bisimulation Z . We define a $\text{LTL} \times \text{S5}$ model $K = (H, \phi, 0)$ such that $K \cong_{\Theta} M$ and $K \cong_{\Lambda} N$ as follows: We let $H = \{a' | a = (\tau_a, \mu_a) \in Z\}$ where for all $i, a'(i) = (\tau_a(i) \setminus \Theta) \cup (\mu_a(i) \setminus \Lambda)$. It follows that the relation $X = \{(a', \tau_a) | a = (\tau_a, \mu_a) \in Z\}$ is a Θ -bisimulation from K to M and $Y = \{(a', \mu_a) | a = (\tau_a, \mu_a) \in Z\}$ is a Λ -bisimulation from K to N . Therefore $\text{LTL} \times \text{S5}$ is safe, and hence preserves existential elimination.

Given that $\text{LTL} \times \text{S5}$ is safe it is easier to encode a tiling problem, since we can assume that existential introduction and elimination are sound.

Lemma 8. *The bisimulation quantified extension of $\text{LTL} \times \text{S5}$ is undecidable.*

This can be shown in a similar manner to Lemma 6. The proof is slightly easier since the X operator can be used to identify vertically adjacent states. For horizontally adjacent states we can again use the step properties and bisimulation quantified atoms to define a formula Tile_F^2 that is satisfiable if and only if F can tile the plane.

5 Conclusion

In this paper we have examined a generalized definition for bisimulation quantified modal logics. We have also introduced the notion of *safe* frames, which satisfy the axioms of existential elimination and existential introduction. We have shown that there are decidable modal logics which are not safe and undecidable. However, the decidability of $\text{QL}_{\text{LTL} \times \text{S5}}$ shows that safety alone is not enough to guarantee that extension by bisimulation quantification will preserve decidability. It is an interesting problem to characterize the modal logics for which are decidable when augmented with bisimulation quantification. While this paper provides several negative results, we hope to soon provide a positive decidability result for a general class of safe modal logics.

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